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Non-Polynomial Spline Method for Solving Nonlinear Two Point Boundary Value Problems

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ABSTRACT

A non-polynomial spline method is formulated for solving nonlinear two-point boundary value problems. The theoretical convergence of the method is investigated and well established. To demonstrate the applicability of the method, some model problems are considered and solved for different values of the mesh size h. Moreover, the accuracy of the method also shown by the means of numerical experimentation and it is observed that the proposed method is more accurate than some methods reported in the literature.

Keywords: Convergence; Nonlinear; Non-polynomial spline

INTRODUCTION

The subject of nonlinear differential equations is well established as part of mathematics and its systematic development goes back to the early days of the development of calculus (Abd-Elhameed et al., 2015). Many problems in science and technology are formulated in linear or nonlinear boundary value problems of the second order ordinary differential equations with various types of boundary conditions as in heat transfer, deformation of beams, heat power transmission, boundary layer theory, and deflection in cables and the modeling of chemical reaction. Numerical simulation in engineering, science and in applied mathematics has become a powerful tool to model the physical phenomena, particularly when analytical solutions are not available or very difficult to obtain analytical solution, (Rahman et al., 2012; Zanariah et al., 2013; Osama et al., 2015; Zurni and Oluwaseun, 2016a; Zurni and Oluwaseun, 2016b).

The general form of second order nonlinear two-point boundary value problems are given in the form of:

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \qquad a \le x \le b \tag{1}$$

subject to the boundary conditions:

$$y(a) = \phi_1 \text{ and } y(b) = \phi_2 \tag{2}$$

where, a, b, ϕ_1 and ϕ_2 are given constants with the unknown function $y(x)$. Most of the time, the boundary value problem of Eq. (1) with one of the boundary conditions of Eq. (2) can be re-written as the governing equation of:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)F(y) = g(x) \tag{3}$$

Where, $p(x), q(x)$ and $g(x)$ are smoothly continuous function for every $x \in [a, b]$.

Here, Eq. (3) may be linear or nonlinear boundary value problems depending on the function $F(y)$. That is, if the function $F(y)$ is linear function of y, then Eq. (3) is linear; otherwise it is nonlinear. Similar to the case of initial value problems, boundary value problems sometimes have more than one solution or the solution may not exist. In recent years, many numerical methods have been developed by different scholars for solving linear and nonlinear second order ordinary differential equations with boundary value problems. For instance: Galerkin method with Hermite polynomials by Rahman et al. (2012); block method by Zanariah et al. (2013); new fourth order quartic spline method by Osama et al. (2015); shooting method by Mizanur et al. (2015); Galerkin residual method using Legendre polynomials by Hossain et al. (2015); five order block method and two-step block method with starting and non-starting values by Zurni and Oluwaseun (2016); and quintic spline method by Osama et al. (2016).

Modeling of the real life phenomena in most cases leads to nonlinear second order ordinary differential equations that may be difficult to solve analytically. Hence, several numerical methods had been developed to solve the nonlinear second order two-point boundary value problems, due to its importance in different areas of the real life phenomena. However, the development of appropriate methods for solving these problems is not exhausted and still demanding for more accuracy and simplicity. Therefore, it is very important to formulate an alternative numerical method that is simple, convergent and more accurate for solving the nonlinear second order two-point boundary value problems.

MATERIAL AND METHOD

Formulation of the Method

Consider the nonlinear second order two-point boundary value problems of the form:

$$y''(x) = f\left(x, y(x), y'(x)\right), \quad a \le x \le b \tag{4}$$

subject to the boundary conditions

$$y(a) = \phi_1, \quad y(b) = \phi_2 . \tag{5}$$

To treat the problem, we first apply the quasi-linearization technique (Bellman and Kalaba, 1965) and linearize Eq. (4). By choosing a reasonable initial approximation for the function $y(x)$ in $f\left(x, y, y'\right)$, call it as $y^{(0)}(x)$ and expand $f\left(x, y, y'\right)$ around the function $y^{(0)}(x)$, we obtain:

$$f\left(x, y^{(1)}, y'^{(1)}\right) = f(x, y^{(0)}, y'^{(0)}) + \left(y^{(1)} - y^{(0)}\right)\left(\frac{\partial f}{\partial y}\right)_{(x, y^{(0)}, y'^{(0)})}$$
$$+\left(y'^{(1)} - y'^{(0)}\right)\left(\frac{\partial f}{\partial y'}\right)_{(x, y^{(0)}, y'^{(0)})} + \cdots$$

In general, we can write for $k = 0,1,2,\cdots$ (k is iteration index),

$$f\left(x, y^{(k+1)}, y'^{(k+1)}\right) = f(x, y^{(k)}, y'^{(k)}) + \left(y^{(k+1)} - y^{(k)}\right)\left(\frac{\partial f}{\partial y}\right)_{(x, y^{(k)}, y'^{(k)})} + \left(y'^{(k+1)} - y'^{(k)}\right)\left(\frac{\partial f}{\partial y'}\right)_{(x, y^{(k)}, y'^{(k)})} + \cdots \tag{6}$$

Thus, after the linearization process Eqs. (4) and (5) becomes:

$$y''^{\,(k+1)}(x) - \left(\frac{\partial f^{(k)}}{\partial y'}\right) y'^{\,(k+1)}(x) - \left(\frac{\partial f^{(k)}}{\partial y}\right) y^{(k+1)}$$
$$= f^{(k)} - y^{(k)}\left(\frac{\partial f^{(k)}}{\partial y}\right) - y'^{(k)}\left(\frac{\partial f^{(k)}}{\partial y'}\right), \tag{7}$$

for $k = 0,1,2,\cdots$, where

$$f^{(k)} = f(x, y^{(k)}, y'^{(k)}), \quad \frac{\partial f^{(k)}}{\partial y} = \left(\frac{\partial f}{\partial y}\right)_{(x, y^{(k)}, y'^{(k)})}, \quad \frac{\partial f^{(k)}}{\partial y'} = \left(\frac{\partial f}{\partial y'}\right)_{(x, y^{(k)}, y'^{(k)})}$$

With

$$y^{(k+1)}(a) = \phi_1 \text{ and } y^{(k+1)}(b) = \phi_2 . \tag{8}$$

This implies that the non-linear differential equation in Eqs. (4) and (5) becomes linear in $y^{(k+1)}$. Solving sequence of linear differential equation given by Eqs. (7) and (8), we can get the approximate solution to the original problem in Eqs. (4) and (5).

Now, in order to develop the non-polynomial spline approximation for the second-order type boundary value problem in Eqs. (7) and (8), the interval $[a,b]$ is divided into N equal sub-intervals. For this, we introduce the set of grid points $x_i = x_0 + ih, \; i = 0,1,2,...,N$, so that,

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b \text{, where } h = \frac{b-a}{N}.$$

Let $y^{(k+1)}(x)$ be the exact solution of the Eqs. (4) and (5) and $y_i^{(k+1)}$ be an approximation to $y^{(k+1)}(x_i)$, obtained by the segment $S_\Delta(x)$ of the spline function passing through the points $(x_i, y_i^{(k+1)})$ and $(x_{i+1}, y_{i+1}^{(k+1)})$ from Eqs. (7) and (8) at $(k+1)^{th}$ iteration. For each i^{th} segment, let us consider the non-polynomial spline function $S_\Delta(x)$ in subinterval $[x_i, x_{i+1}], \; i = 0,1,2,\cdots,N-1$ of the form:

$$\begin{aligned} S_\Delta(x) &= a_i \sin w(x-x_i) + b_i \cos w(x-x_i) + \\ &\quad c_i\left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right) + d_i\left(e^{w(x-x_i)} + e^{-w(x-x_i)}\right) \end{aligned} \tag{9}$$

Where, a_i, b_i, c_i and d_i are constants to be determined and $w \neq 0$ is a parameter which is used to raise the accuracy of the method.
To determine the unknown coefficient in Eq. (9), let us denote:

$$S_\Delta(x_i) = y_i^{(k+1)}, \quad S_\Delta(x_{i+1}) = y_{i+1}^{(k+1)}, \quad S''_\Delta(x_i) = M_i, \quad S''_\Delta(x_{i+1}) = M_{i+1} \tag{10}$$

Differentiating Eq. (9) successively, we get:

$$\begin{aligned} S'_\Delta(x) &= a_i w \cos w(x-x_i) - b_i w \sin w(x-x_i) + \\ &\quad c_i w\left(e^{w(x-x_i)} + e^{-w(x-x_i)}\right) + d_i w\left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right) \end{aligned} \tag{11}$$

$$\begin{aligned} S''_\Delta(x) &= -a_i w^2 \sin w(x-x_i) - b_i w^2 \cos w(x-x_i) + \\ &\quad c_i w^2\left(e^{w(x-x_i)} - e^{-w(x-x_i)}\right) + d_i w^2\left(e^{w(x-x_i)} + e^{-w(x-x_i)}\right) \end{aligned} \tag{12}$$

Using relations in Eqs. (10) and (12), we have:

$$S''(x_i) = M_i = -b_i w^2 + 2d_i w^2$$

which in turn resulted in

$$2d_i = b_i + \frac{M_i}{w^2} \tag{13}$$

Again, using the relation in Eq. (10), Eq. (13) into Eq. (9) at the point x_i, we obtain:

$$S(x_i) = y_i^{(k+1)} = b_i + 2d_i$$

and it gives

$$b_i = \frac{w^2 y_i^{(k+1)} - M_i}{2w^2} \tag{14}$$

Substituting Eq. (14) into Eq. (13), we get:

$$d_i = \frac{M_i + w^2 y_i^{(k+1)}}{4w^2} \tag{15}$$

Using the relation in Eq. (10), Eq. (14) in Eq. (12) at the point x_{i+1} and letting $wh = \theta$, we have:

$$\frac{M_{i+1}}{w^2} = -a_i \sin\theta + \left(\frac{M_i - w^2 y_i^{(k+1)}}{2w^2} \right) \cos\theta + c_i \left(e^{\theta} - e^{-\theta}\right) + d_i \left(e^{\theta} + e^{-\theta}\right) \tag{16}$$

Using the relation in Eq. (10), Eq. (14) in Eq. (9) at the point x_{i+1}, we have:

$$y_{i+1}^{(k+1)} = a_i \sin\theta + \left(\frac{w^2 y_i^{(k+1)} - M_i}{2w^2} \right) \cos\theta + c_i \left(e^{\theta} - e^{-\theta}\right) + d_i \left(e^{\theta} + e^{-\theta}\right) \tag{17}$$

Subtracting Eq. (16) from Eq. (17), we obtain:

$$a_i = \frac{w^2 y_{i+1}^{(k+1)} - M_{i+1} + (M_i - w^2 y_i^{(k+1)}) \cos\theta}{2w^2 \sin\theta} \tag{18}$$

Adding Eq. (16) and (17), we obtain:

$$\frac{w^2 y_{i+1}^{(k+1)} + M_{i+1}}{w^2} = 2c_i \left(e^{\theta} - e^{-\theta}\right) + 2d_i \left(e^{\theta} + e^{-\theta}\right) \tag{19}$$

Substituting Eq. (15) into Eq. (19), we get:

$$c_i = \frac{2\left(w^2 y_{i+1}^{(k+1)} + M_{i+1}\right) - \left(M_i + w^2 y_i^{(k+1)}\right)\left(e^{\theta} + e^{-\theta}\right)}{4w^2 \left(e^{\theta} - e^{-\theta}\right)} \tag{20}$$

Using the continuity condition of the first derivative at x_i, that is $S'_{\Delta-1}(x_i) = S'_{\Delta}(x_i)$, we have,

$$a_{i-1} \cos\theta - b_{i-1} \sin\theta + c_{i-1} \left(e^{\theta} + e^{-\theta}\right) + d_{i-1} w \left(e^{\theta} - e^{\theta}\right) = a_i + 2c_i \tag{21}$$

Reducing indices of Eqs. (14), (15), (18) and (20) by one and substituting into Eq. (21) and simplifying, we obtain:

$$\left(\frac{-1}{2\sin\theta}-\frac{1}{(e^{\theta}-e^{-\theta})}\right)y_{i-1}^{(k+1)}+\left(\frac{\cos\theta}{\sin\theta}+\frac{e^{\theta}+e^{-\theta}}{(e^{\theta}-e^{-\theta})}\right)y_{i}^{(k+1)}$$

$$+\left(\frac{-1}{2\sin\theta}-\frac{1}{(e^{\theta}-e^{-\theta})}\right)y_{i+1}^{(k+1)}=\left(\frac{-h^2}{2\theta^2\sin\theta}+\frac{h^2}{\theta^2(e^{\theta}-e^{-\theta})}\right)M_{i-1}$$

$$+\left(\frac{h^2\cos\theta}{\theta^2\sin\theta}-\frac{h^2(e^{\theta}+e^{-\theta})}{\theta^2(e^{\theta}-e^{-\theta})}\right)M_i$$

$$+\left(\frac{-h^2}{2\theta^2\sin\theta}+\frac{h^2}{\theta^2(e^{\theta}-e^{-\theta})}\right)M_{i+1}$$

which implies

$$y_{i-1}^{(k+1)}-\rho y_i^{(k+1)}+y_{i+1}^{(k+1)}=h^2\left(\alpha M_{i-1}+\beta M_i+\alpha M_{i+1}\right) \tag{22}$$

where,

$$\alpha=\frac{e^{\theta}-e^{-\theta}-2\sin\theta}{\theta^2\left(2\sin\theta+e^{\theta}-e^{-\theta}\right)},$$

$$\beta=\frac{-2\left\{e^{\theta}(\cos\theta-\sin\theta)-e^{-\theta}(\sin\theta+\cos\theta)\right\}}{\theta^2\left(2\sin\theta+e^{\theta}-e^{-\theta}\right)},$$

$$\rho=\frac{2\left\{e^{\theta}(\cos\theta+\sin\theta)+e^{-\theta}(\sin\theta-\cos\theta)\right\}}{2\sin\theta+e^{\theta}-e^{-\theta}}.$$

As $\theta\to 0$, $(\alpha,\beta,\rho)\to\left(\frac{1}{6},\frac{4}{6},2\right)$ and then spline relation defined in Eq. (22) reduces to a polynomial cubic spline relation in Millar et al. (1996).

Now rewriting Eq. (7) and evaluating at the nodal points x_i , we have:

$$y_i''^{(k+1)}=p_i^{(k)}y_i'^{(k+1)}+q_i^{(k)}y_i^{(k+1)}+r_i^{(k)} \tag{23}$$

where

$$p_i^{(k)}=\frac{\partial f_i^{(k)}}{\partial y'},\quad q_i^{(k)}=\frac{\partial f_i^{(k)}}{\partial y}\ \text{ and } r_i^{(k)}=f_i^{(k)}-y_i^{(k)}\left(\frac{\partial f_i^{(k)}}{\partial y}\right)-y_i'^{(k)}\left(\frac{\partial f_i^{(k)}}{\partial y'}\right).$$

Using spline's second derivatives in Eq. (10) at $(k+1)^{th}$ iteration, we have:

$$M_j^{(k+1)}=p_j^{(k)}y_j'^{(k+1)}+q_j^{(k)}y_j^{(k+1)}+r_j^{(k)}\ \text{ for }\ j=i-1,\ i,\ j+1 \tag{24}$$

Using the following approximations for the first derivative of $y_i^{(k+1)}$, (Bawa, 2005):

$$y_{i-1}'^{(k+1)}\cong\frac{-y_{i+1}^{(k+1)}+4y_i^{(k+1)}-3y_{i-1}^{(k+1)}}{2h}, \tag{25}$$

$$y_{i+1}'^{(k+1)}\cong\frac{3y_{i+1}^{(k+1)}-4y_i^{(k+1)}+y_{i-1}^{(k+1)}}{2h}, \tag{26}$$

$$y_i'^{(k+1)} \cong \left\{\frac{1+2\tau h^2 q_{i+1}^{(k)} + \tau h(3p_{i+1}^{(k)} + p_{i-1}^{(k)})}{2h}\right\} y_{i+1}^{(k+1)} - 2\tau(p_{i+1}^{(k)} + p_{i-1}^{(k)}) y_i^{(k+1)}$$
$$- \left\{\frac{1+2\tau h^2 q_{i-1}^{(k)} - \tau h(p_{i+1}^{(k)} + 3p_{i-1}^{(k)})}{2h}\right\} y_{i-1}^{(k+1)} + \tau h(r_{i+1}^{(k)} - r_{i-1}^{(k)}) \quad (27)$$

where τ is a parameter used for increasing the accuracy of the computed solution.

Remark 1: If $\rho = 2$ and $\tau = 0$, the present method reduces to the method of Aziz and Khan (2002).

Substituting Eqs. (24) - (27) into Eq. (22) and rearranging, we obtain the three-term recurrence relation of the form:

$$E_i^{(k)} y_{i-1}^{(k+1)} - F_i^{(k)} y_i^{(k+1)} + G_i^{(k)} y_{i+1}^{(k+1)} = H_i^{(k)}, \; i = 1, 2, \cdots N-1 \quad (28)$$

Where,

$$E_i^{(k)} = 1 + \frac{3}{2}\alpha h p_{i-1}^{(k)} - \alpha h^2 q_{i-1}^{(k)} + \frac{1}{2}\beta h p_i^{(k)} + \beta\tau h^3 p_i^{(k)} q_{i-1}^{(k)}$$
$$- \frac{1}{2}\beta\tau h^2 p_i^{(k)}\left(p_{i+1}^{(k)} + 3p_{i-1}^{(k)}\right) - \frac{1}{2}\alpha h p_{i+1}^{(k)}$$

$$F_i^{(k)} = \rho + 2\alpha h p_{i-1}^{(k)} - 2\beta\tau h^2 p_i^{(k)}\left(p_{i+1}^{(k)} + p_{i-1}^{(k)}\right) + \beta h^2 q_i^{(k)} - 2\alpha h p_{i+1}^{(k)}$$

$$G_i^{(k)} = 1 + \frac{1}{2}\alpha h p_{i-1}^{(k)} - \frac{1}{2}\beta h p_i^{(k)} - \beta\tau h^3 p_i^{(k)} q_{i+1}^{(k)} - \frac{1}{2}\beta\tau h^2 p_i^{(k)}\left(3p_{i+1}^{(k)} + p_{i-1}^{(k)}\right)$$
$$- \frac{3}{2}\alpha h p_{i+1}^{(k)} - \alpha h^2 q_{i+1}^{(k)}$$

$$H_i^{(k)} = h^2\left\{\left(\alpha - \beta\tau h p_i^{(k)}\right) r_{i-1}^{(k)} + \beta r_i^{(k)} + \left(\alpha + \beta\tau h p_i^{(k)}\right) r_{i+1}^{(k)}\right\}$$

Using discrete imbedding algorithm, the tri-diagonal system in Eq (28) in order to obtain the approximations $y_1^{(k+1)}, y_2^{(k+1)}, \cdots, y_{N-1}^{(k+1)}$ of the solution $y(x)$ at $x_1, x_2, \cdots, x_{N-1}$.

Convergence Analysis

In this section, we investigate the convergence analysis of the proposed method. For this, let $\overline{Y}^{(k+1)} = y^{(k+1)}(x_i)$, $Y^{(k+1)} = y_i^{(k+1)}$, $D = d_i$, $T = t_i$, $E = e_i = \overline{Y}^{(k+1)} - Y^{(k+1)}$ for $i = 1, 2, \cdots, N-1$, where $\overline{Y}^{(k+1)}, Y^{(k+1)}, T$ and E are exact solution, approximate solution, local truncation error and discretization error respectively at $(k+1)^{th}$ iteration.

Putting the tri-diagonal system in Eq. (28) in matrix vector form:

$$\left(A + B + h^2 C\right) Y^{(k+1)} = D \quad (29)$$

where

$$A=\begin{bmatrix} -\rho & 1 & 0 & \dots & & 0 \\ 1 & -\rho & 1 & \dots & & 0 \\ 0 & 1 & -\rho & 1 & & \\ \vdots & & & \ddots & & \vdots \\ & & & 1 & -\rho & 1 \\ 0 & \dots & & & 1 & -\rho \end{bmatrix},$$

$$C=\begin{bmatrix} M_1 & K_1 & 0 & \dots & & 0 \\ L_2 & M_2 & K_2 & \dots & & 0 \\ 0 & L_3 & M_3 & K_3 & & \\ \vdots & & & \ddots & & \vdots \\ & & & L_{N-2} & M_{N-2} & K_{N-2} \\ 0 & \dots & & & L_{N-1} & M_{N-1} \end{bmatrix} \text{ and}$$

$$B=\begin{bmatrix} hV_1 & hW_1-h^3Z_1 & 0 & \dots & & 0 \\ hU_2+h^3X_2 & hV_2 & hW_2-h^3Z_2 & \dots & & 0 \\ 0 & hU_3+h^3X_3 & hV_3 & hW_3-h^3Z_3 & & \\ \vdots & & \ddots & & & \vdots \\ & & & hU_{N-2}+h^3X_{N-2} & hV_{N-2} & hW_{N-2}-h^3Z_{N-2} \\ 0 & \dots & & & hU_{N-1}+h^3X_{N-1} & hV_{N-1} \end{bmatrix}$$

are of order $(N-1)$, where

$$U_i=\frac{1}{2}\left(3\alpha p_{i-1}^{(k)}+\beta p_i^{(k)}-\alpha p_{i+1}^{(k)}\right), \quad V_i=2\alpha(p_{i+1}^{(k)}-p_{i-1}^{(k)}), \quad W_i=\frac{1}{2}\left(\alpha p_{i-1}^{(k)}-\beta p_i^{(k)}-3\alpha p_{i+1}^{(k)}\right)$$

$$L_i=-\alpha q_{i-1}^{(k)}-\frac{1}{2}\beta\tau p_i^{(k)}\left(p_{i+1}^{(k)}+3p_{i-1}^{(k)}\right), \qquad M_i=2\beta\tau p_i^{(k)}\left(p_{i+1}^{(k)}+p_{i-1}^{(k)}\right)-\beta q_i^{(k)},$$

$$L_i=-\alpha q_{i+1}^{(k)}-\frac{1}{2}\beta\tau p_i^{(k)}\left(3p_{i+1}^{(k)}+p_{i-1}^{(k)}\right), \qquad X_i=\beta\tau p_i^{(k)}q_{i-1}^{(k)}, \qquad Z_i=\beta\tau p_i^{(k)}q_{i+1}^{(k)}$$

and the $(N-1)$column vector $D=(d_i)$ is given by:

$$d_i=\begin{cases} H_1^{(k)}-E_1^{(k)}\phi_1, & i=1 \\ H_i^{(k)}, & 2\le i\le N-2 \\ H_{N-1}^{(k)}-G_{N-1}^{(k)}\phi_2, & i=N-1 \end{cases}.$$

Now, considering the above system with exact solution $\bar{Y}^{(k+1)}=\left[y^{(k+1)}(x_1),\ y^{(k+1)}(x_2),\ \cdots,\ y^{(k+1)}(x_{N-1})\right]^T$, we have,

$$\left(A+B+h^2C\right)\bar{Y}^{(k+1)}=T(h)+D \qquad (30)$$

where $T(h)=\left[t_1(h),\ t_2(h),\ \cdots,\ t_{N-1}(h)\right]^T$ is the local truncation error associated with the scheme in Eq. (28) which is given by,

$$t_i(h) = -(2-\rho)y_i + (2\alpha+\beta-1)h^2 y''^{(k+1)}(x_i)$$
$$+ h^4\left\{\left(\frac{2}{3}\alpha - \beta\left(\frac{1}{6}+2\tau\right)\right)p^{(k)}(x_i)y'''^{(k+1)}(x_i) + \left(\alpha - \frac{1}{12}\right)y^{(4)\,(k+1)}(x_i)\right\} \quad (31)$$
$$+ h^6\left\{\left(\frac{\alpha}{15} - \beta\left(\frac{\tau}{3}+\frac{1}{120}\right)\right)p^{(k)}(x_i)y^{(5)\,(k+1)}(\xi_i) + \left(\frac{\alpha}{12} - \frac{1}{360}\right)y^{(6)\,(k+1)}(\xi_i)\right\}$$

for $x_{i-1} < \xi_i < x_{i+1}$.

Thus, for different values of α, β, ρ and τ in the scheme of Eq. (28), the following different orders of the method can be obtained.

i. For any choice of arbitrary α and β with $2\alpha+\beta=1$, $\rho=2$ and for any value of τ, the scheme of Eq. (28) gives the second-order method.

ii. For choice $\alpha=\frac{1}{12}$, $\beta=\frac{10}{12}$, $\rho=2$ and $\tau=-\frac{1}{20}$, the scheme of Eq. (28) gives the fourth-order method, and Eq. (31) becomes:

$$t_i(h) = h^6\left\{\frac{p^{(k)}(x_i)}{80}y^{(5)\,(k+1)}(\xi_i) + \frac{1}{240}y^{(6)\,(k+1)}(\xi_i)\right\} \quad (32)$$

Subtracting Eq. (29) from Eq. (30), we get:

$$PE = T(h) \quad (33)$$

where $P = A+B+h^2C$ and $E = \bar{Y}^{(k+1)} - Y^{(k+1)} = \left[e_1, e_2, \cdots, e_{N-1}\right]^T$.

Let S_i be the i^{th} row sum of the matrix P, then we have:

$$\begin{aligned} S_1 &= 1-\rho+\frac{h}{2}\left(\alpha p_{i+1}^{(k)} - \beta p_i^{(k)} - 3\alpha p_{i-1}^{(k)}\right) \\ &+ h^2\left(\frac{\beta\tau p_i^{(k)}}{2}(p_{i+1}^{(k)} + 3p_{i-1}^{(k)}) - \beta q_i^{(k)} - \alpha q_{i+1}^{(k)}\right) + O(h^3) \end{aligned}, \quad i=1 \quad (34)$$

$$S_i = 2-\rho-h^2\left(\alpha q_{i-1}^{(k)} + \beta q_i^{(k)} + \alpha q_{i+1}^{(k)}\right) + O(h^3), \quad i = 2,3,\cdots,N-2 \quad (35)$$

$$\begin{aligned} S_{N-1} &= 1-\rho+\frac{h}{2}\left(3\alpha p_{i+1}^{(k)} + \beta p_i^{(k)} - \alpha p_{i-1}^{(k)}\right) + \\ & h^2\left(\frac{\beta\tau p_i^{(k)}}{2}(3p_{i+1}^{(k)} + p_{i-1}^{(k)}) - \beta q_i^{(k)} - \alpha q_{i-1}^{(k)}\right) + O(h^3) \end{aligned}, \quad i = N-1 \quad (36)$$

For sufficiently small h, the matrix P is monotone, (Young, 1971). It follows that P^{-1} exists and its elements are nonnegative.

Hence, from Eq. (33), we have:

$$E = P^{-1}T(h) \quad \text{and} \quad \|E\| \le \|P^{-1}\|\|T(h)\| \quad (37)$$

Let $\bar{p}_{i,j}$ be the $(i,j)^{th}$ element of the matrix P^{-1} and define,

$$\left\|P^{-1}\right\| = \max_{1\le i\le N-1}\sum_{j=1}^{N-1}\bar{p}_{i,j} \quad \text{and} \quad \|T(h)\| = \max_{1\le i\le N-1}\left|t_i(h)\right| \tag{38}$$

Since $\bar{p}_{i,j} \ge 0$, from the properties of monotone and non-negative matrices, we have:

$$\sum_{j=1}^{N-1}\bar{p}_{i,j}S_j = 1, \quad \text{for} \quad i = 1,2,\cdots,N-1 \tag{39}$$

Hence, from Eqs. (34) – (36), we have:

$$\bar{p}_{i,1} \le \frac{1}{S_1} < \frac{1}{h^2Q(\alpha+\beta)}, \quad \text{for } j = 1 \tag{40}$$

$$\bar{p}_{i,N-1} \le \frac{1}{S_{N-1}} < \frac{1}{h^2Q(\alpha+\beta)}, \text{ for } j = N-1 \tag{41}$$

Further, $\displaystyle\sum_{j=2}^{N-2}\bar{p}_{i,j} \le \frac{1}{\min\limits_{2\le j\le N-2} S_j} < \frac{1}{h^2Q(2\alpha+\beta)}$, (42)

where, $Q = \min\limits_{1\le i\le N}\left|q_i^{(k)}\right| = \min\limits_{1\le i\le N}\left|-q_i^{(k)}\right|$.

Hence, from Eqs. (40) – (42), we obtain:

$$\sum_{j=1}^{N-1}\bar{p}_{i,j} \le \frac{1}{\min\limits_{1\le j\le N-1} S_j} < \frac{1}{h^2Q(4\alpha+3\beta)} \tag{43}$$

From Eqs. (32), (37), (38) and (43), we get:

$$\|E\| \le \frac{1}{240h^2Q(4\alpha+3\beta)}h^6\left\{3\left|p^{(k)}(x_i)y^{(5)\,(k+1)}(\xi_i)\right| + \left|y^{(6)\,(k+1)}(\xi_i)\right|\right\} = C^*h^4$$

Where, $C^* = \dfrac{1}{240Q(4\alpha+3\beta)}\left\{3\left|p^{(k)}(x_i)y^{(5)\,(k+1)}(\xi_i)\right| + \left|y^{(6)\,(k+1)}(\xi_i)\right|\right\}$, which is independent on mesh size h.

Hence, the method in Eq. (28) is fourth-order convergent for $\alpha = \dfrac{1}{12}$, $\beta = \dfrac{10}{12}$, $\rho = 2$ and $\tau = -\dfrac{1}{20}$.

Theorem: The method given by Eq. (28) for solving the boundary value problem of Eqs. (4) and (5) for sufficiently small h gives a fourth order convergent solution.

RESULT AND DISCUSSION

In order to test the validity of the proposed method and to demonstrate its convergence computationally, we have considered five model examples of nonlinear second order two-point boundary value problems with exact solutions. The maximum absolute errors at the nodal points are given by,

$\|E\|_\infty = \max_{1\le i\le N-1} |y(x_i) - y_i|$ Where $y(x_i)$ and y_i are exact solution and numerical solution respectively, at the nodal point x_i.

The numerical results are tabulated in Tables (1) - (4) and depicted in Figures (1) - (5). The computed solutions are compared with the exact solutions at nodal points and compared with the methods in (Saeed and Rehman, 2014; Perfilieva et al., 2017; Pandey, 2018).

Remark 2: All numerical results of Examples 1 - 5 are solved for the second iteration of quasi-linearization method.

Example 1: Consider the nonlinear second order differential equation:

$$y''(x) + y'(x) + y^3(x) + y(x) = f(x) \tag{44}$$

subject to the boundary conditions $y(0)=0, \quad y(1)=0$, where

$$f(x) = 2 + 2x + x^2 - 20x^3 - 5x^4 - x^5 + x^6 - 3x^9 + 3x^{12} - x^{15}$$

The analytical solution of this problem is $y(x) = x^2 - x^5$, (Saeed and Rehman, 2014).

Applying the quasi-linearization technique to Eq. (44), we get:

$$y''^{(k+1)}(x) + y'^{(k+1)}(x) + \left(1 + 3(y^{(k)}(x))^2\right) y^{(k+1)}(x) = f(x) + 2(y^{(k)}(x))^3$$

with the boundary conditions $y^{(k+1)}(0)=0, \quad y^{(k+1)}(1)=0$.

Table 1. Pointwise absolute errors of Example 1 with different values of mesh size *h*.

	Saeed and Rehman, (2014)	Present Method		
x		$N=10$	$N=20$	$N=30$
0.1	$4.2051152889e-05$	1.0271e-05	5.7289e-07	5.3917e-08
0.2	$8.1722744304e-05$	1.7961e-05	9.9183e-07	8.3736e-08
0.3	$1.1640750572e-04$	2.3165e-05	1.2636e-06	9.1560e-08
0.4	$1.4175454098e-04$	2.5988e-05	1.3979e-06	8.1994e-08
0.5	$1.5166050983e-04$	2.6547e-05	1.4084e-06	6.3216e-08
0.6	$1.3826276219e-04$	2.4965e-05	1.3117e-06	4.5973e-08
0.7	$9.1944320423e-05$	2.1372e-05	1.1220e-06	3.8417e-08
0.8	$1.3602317094e-06$	1.5908e-05	8.4401e-07	3.7912e-08
0.9	$1.4651915492e-04$	8.7281e-06	4.7154e-07	2.9710e-08
$\|E\|_\infty$	--	2.6547e-05	1.4176e-06	9.1560e-08

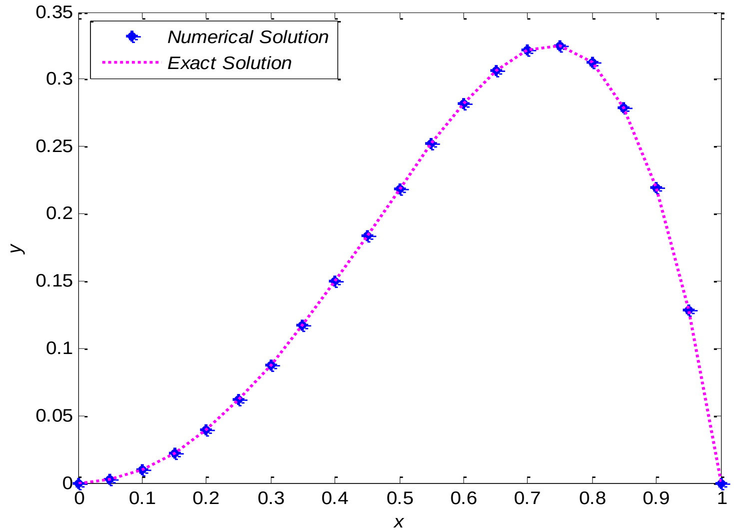

Fig. 1. Numerical solution versus exact solution for Example 1 when $N = 20$.

Example 2: Consider the nonlinear second order differential equation:

$$y''(x) = \frac{1}{2}(1 + y + x)^3 \tag{45}$$

subject to the boundary conditions $y\left(\frac{1}{2}\right) = -\frac{1}{6}, \quad y(1) = 0$.

The analytical solution of this problem is $y(x) = \dfrac{2}{2-x} - x - 1$, (Pandey, 2018).

Applying the quasi-linearization technique to Eq. (45), we get:

$$y''^{(k+1)}(x) - \frac{3}{2}\left(1 + x + y^{(k)}(x)\right)^2 y^{(k+1)}(x)$$
$$= \frac{1}{2}\left(1 + x + y^{(k)}(x)\right)^3 - \frac{3}{2}\left(1 + x + y^{(k)}(x)\right)^2 y^{(k)}(x)$$

with the boundary conditions $y^{(k+1)}\left(\frac{1}{2}\right) = -\frac{1}{6}, \quad y^{(k+1)}(1) = 0$.

Table 2. Point wise absolute errors of Example 2 with different values of mesh size *h*.

	Present Method			
x	$N = 4$	$N = 8$	$N = 16$	$N = 32$
0.5625	-	2.1021e-07	1.2401e-08	1.8137e-11
0.625	6.2310e-06	3.9454e-07	2.3224e-08	9.1213e-11
0.6875	-	5.4528e-07	3.2102e-08	1.2342e-10
0.75	1.0258e-05	6.5020e-07	3.8425e-08	3.4894e-12
0.8125	-	6.8985e-07	4.1077e-08	3.2565e-10

0.875	9.9676e-06	6.3319e-07	3.8082e-08	6.9484e-10
0.9375	-	4.3031e-07	2.6150e-08	7.5380e-10
$\|E\|_\infty$	1.0258e-05	6.8985e-07	4.1077e-08	7.9395e-10
$\|E\|_\infty$, **Pandey, (2018)**	0.7207642e-03	0.1922799e-03	0.9181426e-04	0.8538120e-04

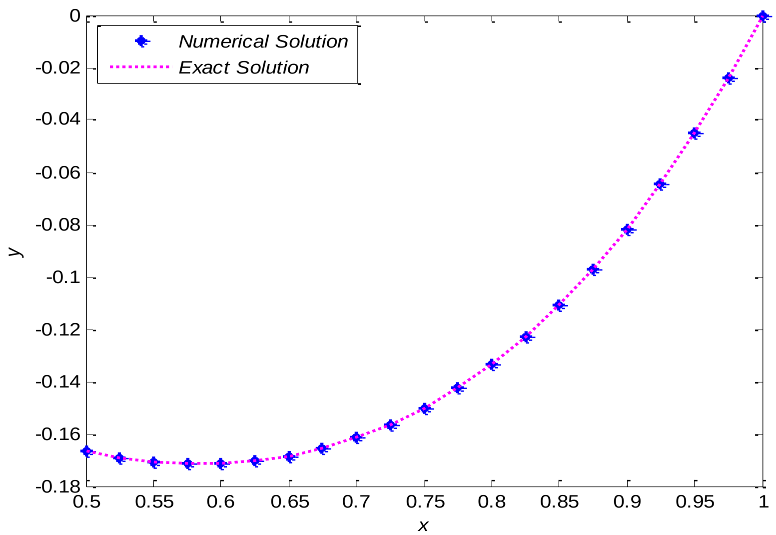

Fig. 2. Numerical solution versus exact solution for Example 2 when $N = 20$.

Example 3: Consider the nonlinear second order differential equation:

$$y''(x) = 2y(x)y'(x) \tag{46}$$

subject to the boundary conditions $y(0)=0,\quad y(\pi/4)=1$,

The analytical solution of this problem is $y(x)=\tan x$, (Perfilieva et al., 2017).
Applying the quasilinearization technique to Eq. (46), we get:

$$y''^{(k+1)}(x)-2y^{(k)}(x)y'^{(k+1)}(x)-2y'^{(k)}(x)y^{(k+1)}(x)=-2y^{(k)}(x)y'^{(k)}(x)$$

with the boundary conditions $y^{(k+1)}(0)=0,\quad y^{(k+1)}(\pi/4)=1$.

Table 3. Pointwise absolute errors of Example 3 with different values of mesh size *h*.

x	$N = 20$	$N = 30$	$N = 40$
$\pi/40$	1.3208e-08	1.2160e-08	5.9318e-09
$\pi/20$	1.3452e-08	1.1378e-08	9.4852e-10
$(3\pi)/40$	2.4125e-08	5.9883e-09	2.9370e-09
$\pi/10$	5.4773e-08	3.1343e-08	2.7401e-08
$\pi/8$	7.7772e-08	4.9909e-08	4.5221e-08
$(3\pi)/20$	8.3113e-08	5.2215e-08	4.7016e-08
$(7\pi)/40$	7.3376e-08	4.1704e-08	3.6376e-08

$\pi/5$	5.8112e-08	2.9350e-08	2.4512e-08
$(9\pi)/40$	3.7901e-08	1.8190e-08	1.4875e-08

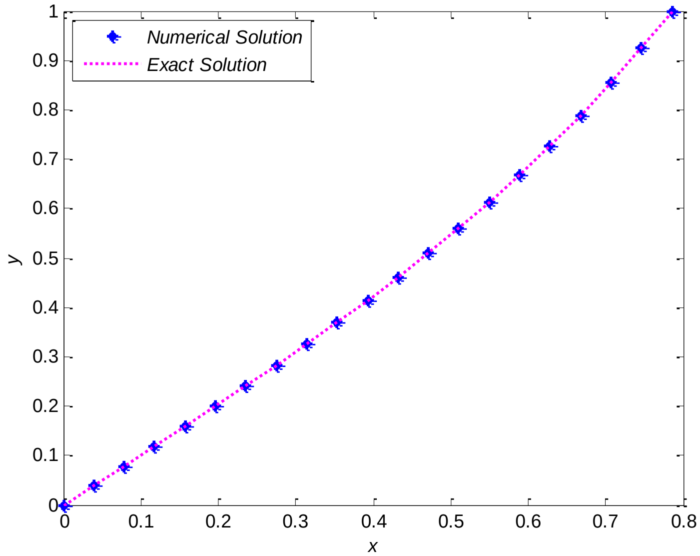

Fig. 3. Numerical solution versus exact solution for Example 3 when $N = 20$.

Example 4: Consider the nonlinear second order differential equation:

$$y''(x) = y^2(x) + 2\pi^2 \cos(2\pi x) - \sin^4(\pi x) \quad (47)$$

subject to the boundary conditions $y(0) = 0,\quad y(1) = 0$,

The analytical solution of his problem is $y(x) = \sin^2(\pi x)$, (Perfilieva et al., 2017).

Applying the quasilinearization technique to Eq. (47), we get:

$$y''^{(k+1)}(x) - 2y^{(k)}(x)y^{(k+1)}(x) = -(y^{(k)}(x))^2 + 2\pi^2 \cos(2\pi x) - \sin^4(\pi x)$$

with boundary conditions $y^{(k+1)}(0) = 0,\quad y^{(k+1)}(1) = 0$.

Table 4. Pointwise absolute errors of Example 4 with different values of mesh size *h*.

x	$N = 20$	$N = 30$	$N = 100$	$N = 200$
0.1	1.0437e-04	1.0238e-04	1.0189e-04	1.0189e-04
0.2	2.1128e-04	2.0223e-04	2.0003e-04	2.0001e-04
0.3	3.0794e-04	2.8982e-04	2.8541e-04	2.8537e-04
0.4	3.7685e-04	3.5133e-04	3.4511e-04	3.4507e-04
0.5	4.0203e-04	3.7367e-04	3.6677e-04	3.6672e-04
0.6	3.7685e-04	3.5133e-04	3.4511e-04	3.4507e-04
0.7	3.0794e-04	2.8982e-04	2.8541e-04	2.8537e-04
0.8	2.1128e-04	2.0223e-04	2.0003e-04	2.0001e-04
0.9	1.0437e-04	1.0238e-04	1.0189e-04	1.0189e-04

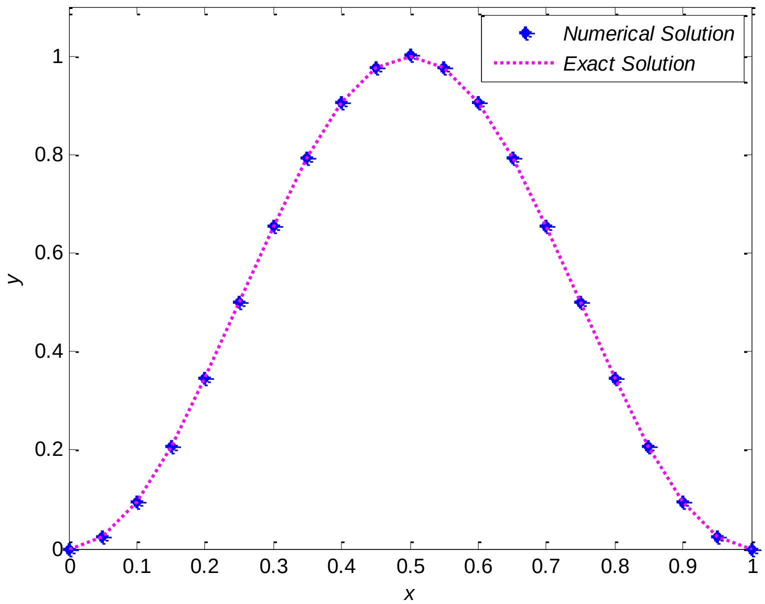

Fig. 4. Numerical solution versus exact solution for Example 4 when $N = 20$.

Example 5: Consider the nonlinear second order differential equation:

$$y''(x) = -\frac{2}{x}\, y(x)\, y'(x) \qquad (48)$$

subject to the boundary conditions $y(1) = 1/2,\ \ y(2) = 2/3$,

The analytical solution of this problem is $y(x) = 1 - \frac{1}{1+x}$, (Perfilieva et al., 2017).

Applying the quasilinearization technique to Eq. (48), we get:

$$y''^{(k+1)}(x) + \frac{2}{x} y^{(k)}(x) y'^{(k+1)}(x) + \frac{2}{x} y'^{(k)}(x) y^{(k+1)}(x) = \frac{2}{x} y^{(k)}(x) y'^{(k)}(x)$$

with boundary conditions $y^{(k+1)}(1) = 1/2,\ \ y^{(k+1)}(2) = 2/3$.

Table 5. Pointwise absolute errors of Example 5 with different values of mesh size *h*.

x	$N = 20$	$N = 30$	$N = 100$	$N = 200$
1.1	4.0443e-06	4.0441e-06	4.0440e-06	4.0440e-06
1.2	6.2073e-06	6.2069e-06	6.2068e-06	6.2068e-06
1.3	6.1187e-06	6.1183e-06	6.1182e-06	6.1182e-06
1.4	4.2301e-06	4.2297e-06	4.2296e-06	4.2296e-06
1.5	1.4333e-06	1.4330e-06	1.4329e-06	1.4329e-06
1.6	1.2707e-06	1.2709e-06	1.2710e-06	1.2710e-06

1.7	3.0302e-06	3.0304e-06	3.0304e-06	3.0304e-06
1.8	3.3338e-06	3.3339e-06	3.3339e-06	3.3339e-06
1.9	2.1458e-06	2.1458e-06	2.1459e-06	2.1459e-06

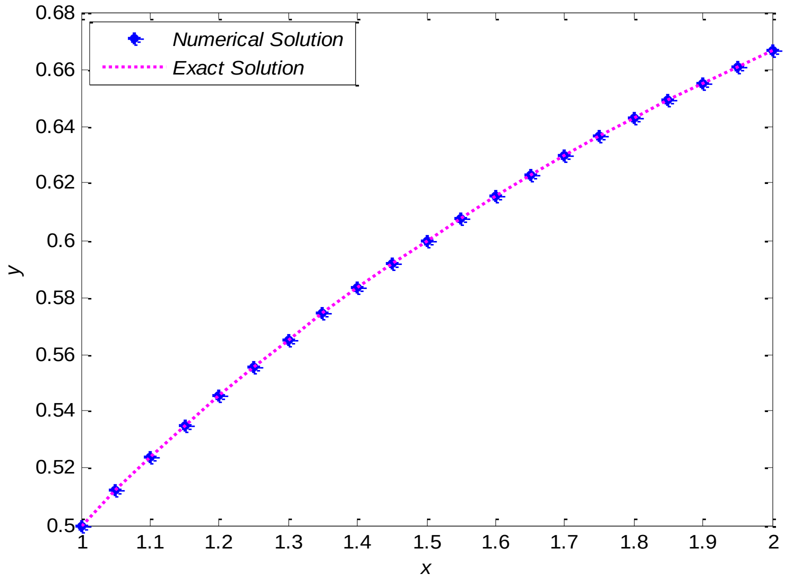

Fig. 5. Numerical solution versus exact solution for Example 5 when $N = 20$.

Table 6. Maximum absolute errors of Examples (Eg.) 3 – 5 with different values of mesh size *h*.

Egs	N	$N = 20$	$N = 30$	$N = 100$	$N = 200$
Eg. 3	Present Method	8.3113e-08	5.3313e-08	4.6024e-08	4.5968e-08
	Perfilieva et al., (2017)	0.0028	-	-	-
Eg. 4	Present Method	4.0203e-04	3.7367e-04	3.6677e-04	3.6672e-04
	Perfilieva et al., (2017)	0.0482	-	-	-
Eg. 5	Present Method	6.4308e-06	6.4183e-06	6.4304e-06	6.4331e-06
	Perfilieva et al., (2017)	9.995e-05	-	-	-

CONCLUSION

In this paper, the non-polynomial spline method is developed to find the approximate solution of nonlinear second order two-point boundary value problems. The technique of quasi-linearization is applied to linearize the nonlinearity of the problems. The convergence analysis of the method is established and it is shown that the present method is of fourth order convergent. To validate the applicability of the proposed method five nonlinear differential problems are considered for different values of mesh size h at the second iteration. The present method gives more accurate results than some methods reported in the document (Tables 1, 2 and 6). Moreover, the absolute errors (*i.e.*, pointwise and maximum) decreases rapidly as N increases, which in turn shows the convergence of the computed solution. Figures (1) - (5) also depicts that the present method approximates the exact solution very well.

In a concise manner, the present method is simple to apply, convergent and more accurate for solving nonlinear second order two-point boundary value problems.

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