FULL-LENGTH ARTICLE

Orthogonal Polynomial Approach to the Solution of Space Fractional Order Wave Equation

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ABSTRACT

In this work, a numerical scheme is proposed to solve Space Fractional Order Wave Equation (SFOWE). This approach uses shifted Chebyshev Polynomials of the fourth kind. The fractional derivatives are expressed in Caputo sense. Thereafter, the Chebyshev collocation method together with finite difference method were used to reduce the system of second order ordinary differential equations so obtained to a system of linear algebraic equations. These are then solved to obtain the unknown constants that are later substituted into the assumed solution to get the desired approximate solution which are presented in tabular form and 3D graphs. Comparison was made between the exact solution and the solution obtained using the proposed method. The results show that the proposed method compared favourably with the existing results in the literature. The approach advocated in the present work does not make use of any linearization procedure, nor does it makes use of implicit numerical scheme, and that makes it attractive than the earlier approaches in the literature.

Keywords: Space Fractional Order, Fourth Kind Chebyshev Polynomial, Finite Difference Method, Collocation Method, Matrices, Gamma Function

INTRODUCTION

The field of fractional calculus is as old as Calculus itself, but in the last few decades, the usefulness of this mathematical theory in application as well as its merit in pure and applied mathematics has become more apparent (Kai & Neville, 2005). Fractional calculus is a field of applied mathematics that deals with the derivatives and integrals of arbitrary orders (Oldham & Spanier, 1974; Miller & Ross, 1993).

In the last few years, there are many studies on fractional differential equations because of their significance in many areas like physics, medicine, and engineering (Hilfer, 2000). The field of fractional calculus allows us to study fractal phenomena which can not be studied by the ordinary case (Liu et al., 2012).

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes (Press et al., 1992). It was also discovered that there are several other applications of the fractional differential equations as can be found in (Diethelm & Ford, 2004; Khader, 2011; Sweilam & Nagy, 2011).

The studied models have received a great attention in the field of viscoelsatic materials (Bagley & Calico, 1999), electrochemical process, (Ichise et al., 1971) control theory (Poulubny, 1999), advection and dispersion of solutes in natural porous or fractured media (Mark & Hall, 1981), signal processing and image filtering (Benson et al., 2000).

Seeking solution of fractional order differential equations is still a tedious task, except in a limited number of these equations. There is difficulty in finding their analytical as well as approximate solutions. Therefore, there have been a lot of attempts by many researchers to develop new methods for obtaining analytical and approximate solutions of these equations. As a result, several methods have been demonstrated for finding the numerical solution of these equations. For example, homotopy Analysis method (Hashim et al., 2009), Adomian decomposition method (Jafari & Daftardar-Gejji, 2006), homotopy perturbation method (Sweilam et al., 2008), collocation method (Tadjeran & Meerschaert, 2007; Khader & Babatin, 2013) and the references cited therein.

Many authors tried to model diffusion and wave equations from the classical diffusion or wave equation by replacing the first or second order space derivative by a fractional derivative of order μ with $0 < \mu < 2$ see (El-Sayed, 1996; Mainardi et al., 2001; Al-Sayed & Aly, 2002), while the use of Gengenbauer polynomial was employed for the solution of the same class of problems (Issa et al., 2022).

This work is focused on the numerical solution of space fractional order wave equation by exploiting the accuracy of shifted Chebyshev Polynomials of the fourth kind. Fourth kind shifted Chebyshev Polynomials together with the fractional derivatives interpreted in Caputo sense is used to transform (SFOWE) into a system of second order ordinary differential equations using Finite Difference Method (FDM). The resulting system of linear algebraic equations are then solved for the unknown constants.

STATEMENT OF THE PROBLEM

The class of problem considered in this work is the space fractional order wave equation of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \sigma(x,t)\frac{\partial^\mu u(x,t)}{\partial x^\mu} + \rho(x,t), \tag{1}$$

on a finite domain $a < x < b$, $0 \le t \le T$. The parameter μ refers to the fractional order of partial derivative with $0 < \mu < 2$. The function $\rho(x,t)$ is a non-homogeneous source term and the coefficient function $\sigma(x,t) \ge 0$. We also assumed that the problem (1) is subject to the following initial conditions:

$$u(x,0) = u^0(x), \quad u_t(x,0) = u'(x) \tag{2}$$

and the Dirichlet boundary conditions

$$u(a,t) = u(b,t) = 0. \tag{3}$$

It is immediately noticeable that the classical wave equation is recovered when $\mu = 2$ in (1), that is

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \sigma(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + \rho(x,t), \tag{4}$$

MATERIALS AND METHODS

The Caputo Fractional Derivative

Let D^γ be the Caputo fractional derivative of order γ which is defined as

$$D^\gamma f(x) = \frac{1}{\Gamma(n-\gamma)}\int_0^x (x-t)^{n-\gamma-1} f^{(n)}(t)dt; \quad \gamma > 0, \tag{5}$$

where $n-1<\gamma<n,\ n\epsilon N,\ \ x>0$.
Like classical differential operator, Caputo derivative also satisfies the linearity property. Thus,

$$D^{\gamma}\big(\xi f(x)+\eta g(x)\big)=\ \xi D^{\gamma}f(x)+\eta D^{\gamma}g(x) \tag{6}$$

where ξ and η are constants. For the Caputo derivative, the following results hold

$$D^{\gamma}x^{p}=\begin{cases}0, & for\ \ p\epsilon N_0\ \ and\ p<\lceil\gamma\rceil\\ \dfrac{\Gamma(p+1)}{\Gamma(\mathrm{p}+1-\gamma)}x^{p-\gamma}, & for\ p\epsilon N_0\ \ and\ p\geq\lceil\gamma\rceil,\end{cases} \tag{7}$$

where the function $\lceil\gamma\rceil$ is the smallest integer greater than or equal to γ and N_0 represents the natural numbers including zero.

Some Properties of Fourth Kind Chebyshev Polynomials

In this section, some excellent properties of the fourth kind Chebyshev polynomials are highlighted. These include its derivation from trigonometric half-angle formula in the interval [-1,1], its shifted form and adaptation to fractional order terms.

Chebyshev Polynomial of Fourth Kind in the Interval [-1, 1] and Its Equivalent in [a, b]

The Chebyshev Polynomials $W_m(x)$ of the fourth kind are orthogonal polynomials of degree m in x defined on [-1, 1], takes the form

$$W_m(x)=\sin(m+\tfrac{1}{2}\alpha)\Big/\sin\tfrac{1}{2}\alpha\,,$$

where $x=\cos\alpha$ and $\alpha\ \epsilon\ [0,\pi]$.

The orthogonality of the fourth kind polynomials is shown as

$$\langle W_m(x),W_p(x)\rangle=\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,W_m(x)W_p(x)dx=\begin{cases}0;m\neq p\\ \pi;m=p\end{cases}, \tag{8}$$

where $\sqrt{\frac{1-x}{1+x}}$ is the weight function in the interval $[-1,1]$. The polynomial is generated through the recurrence relation

$$W_{m+1}(x)=2xW_m(x)-W_{m-1}(x),\ \ m\geq 1, \tag{9}$$

with the initial terms $W_0(x)=1$ and $W_1(x)=2x-1$.

For the general interval $[a,b]$, the shifted Chebyshev polynomial of the fourth kind is obtained as

$$W_{m+1}^{*}(x) = 2\left\{\frac{2x-a-b}{b-a}\right\}W_m(x) - W_{m-1}(x), \;\; m = 1,2,\ldots. \tag{10}$$

For the specific case where $x \epsilon [0,2]$, that is $a = 0$ and $b = 2$, we have the shifted form of (9) as

$$W_{m+1}^{*}(x) = 2(x-1)W_m(x) - W_{m-1}(x), \;\; m \geq 1, \tag{11}$$

Thus, (8) becomes

$$\int_0^2 \sqrt{\frac{2-x}{x}}\, W_m^{*}(x) W_p^{*}(x)\, dx = \begin{cases} 0; m \neq p \\ \pi; m = p \end{cases}, \tag{12}$$

and the corresponding weight is now $\sqrt{\frac{2-x}{x}}$.

The analytical form of the shifted Chebyshev polynomials of the fourth kind $W_m^{*}(x)$ of degree m in x is given by

$$W_m^{*}(x) = \sum_{q=0}^{m} (-1)^q 2^{m-q} \frac{\Gamma(2m-q+1)}{\Gamma(q+1)\Gamma(2m-2q+1)} x^{m-q}, \;\; m \geq 0. \tag{13}$$

The function $r(x)$ which will be required in the solution of (1) can be written as series of W_m^{*} as $r(x) = \sum_{j=0}^{\infty} b_j W_j^{*}(x)$, where the coefficients $b_j,\ j = 0,1,2,\ldots$, are generated by

$$b_j = \frac{1}{\pi}\int_0^2 r(x) \sqrt{\frac{2-x}{x}}\, W_j^{*}(x)\, dx. \tag{14}$$

In practice, only the first $(n+1)$ terms of shifted Chebyshev polynomial of the fourth kind are considered in the approximation. Thus, we have

$$r_n(x) = \sum_{j=0}^{n} b_j W_j^{*}(x). \tag{15}$$

Evaluation of the Fractional Derivatives using Shifted Chebyshev Polynomial of the Fourth Kind

In this section, an approximate formula for the evaluation of fractional derivatives in $r(x)$ is derived. Suppose $r(x)$ is approximated with the aid of the polynomial by taken $\alpha > 0$, and combining all that, with linearity property of Caputo differential operator, we have

$$D^{\alpha}\big(r_n(x)\big) = \sum_{j=0}^{n} b_j D^{\alpha}\left(W_j^{*}(x)\right). \tag{16}$$

It follows from the Caputo derivative given in (7), that

$$D^{\alpha}\left(W_j^*(x)\right) = 0, \qquad j = 0,1,\dots,\lceil \alpha - 1 \rceil, \qquad \alpha > 0 \tag{17}$$

And

$$D^{\alpha}\left(W_j^*(x)\right) = \sum_{q=0}^{j-\lceil \alpha \rceil} (-1)^q 2^{1-q} \frac{\Gamma(2j-q+1)}{\Gamma(q+1)\Gamma(2j-2q+1)} D^{\alpha} x^{j-q} \tag{18}$$

Using (7) and (8), we have

$$D^{\alpha}\left(W_j^*(x)\right) = \sum_{q=0}^{j-\lceil \alpha \rceil} (-1)^q 2^{1-q} \frac{\Gamma(2j-q+1)\Gamma(j-q+1)}{\Gamma(q+1)\Gamma(2j-2q+1)\Gamma(j+1-q-\alpha)} D^{\alpha} x^{j-q} \tag{19}$$

Combining (16), (17) and (19) gives

$$D^{\alpha}\left(r_n(x)\right) = \sum_{j-\lceil \alpha \rceil}^{n} \sum_{q=0}^{j-\lceil \alpha \rceil} b_j (-1)^q 2^{(1-q)} \frac{\Gamma(2j-q+1)\Gamma(j-q+1)}{\Gamma(q+1)\Gamma(2j-2q+1)\Gamma(j+1-q-\alpha)} x^{j-q-\alpha} \tag{20}$$

Which can as well be expressed as

$$D^{\alpha}\left(r_n(x)\right) = \sum_{j-\lceil \alpha \rceil}^{n} \sum_{q=0}^{j-\lceil \alpha \rceil} b_j F_{j,q}^{(\alpha)} x^{j-q-\alpha}, \tag{21}$$

where

$$F_{j.q}^{(\alpha)} = (-1)^q 2^{(1-q)} \frac{\Gamma(2j-q+1)\Gamma(j-q+1)}{\Gamma(q+1)\Gamma(2j-2q+1)\Gamma(j+1-q-\alpha)} x^{j-q-\alpha} \tag{22}$$

To test for the accuracy of our formulae, we consider $r(x) = x^3$ with $n = 4$ and $\alpha = 1.8$.
Using (4), we have

$$D^{1.8}x^3 = \frac{\Gamma(3+1)}{\Gamma(3+1-1.8)} x^{3-1.8} = \frac{\Gamma(4)}{\Gamma(2.2)} x^{1.2} = \frac{6}{\Gamma(\frac{11}{5})} x^{\frac{6}{5}}. \tag{23}$$

Now, using our proposed method given in (18) and (19), we have

$$D^{1.8}x^3 = \sum_{j-2}^{4} \sum_{q=0}^{j-2} b_j F_{j,q}^{(1.8)} x^{j-q-1.8}$$

$$D^{1.8}x^3 = b_2 F_{2,0}^{\frac{9}{5}} x^{\frac{1}{5}} + b_3 F_{3,0}^{\frac{9}{5}} x^{\frac{6}{5}} + b_2 F_{3,1}^{\frac{9}{5}} x^{\frac{1}{5}} + b_4 F_{4,0}^{\frac{9}{5}} x^{\frac{11}{5}} + b_4 F_{4,1}^{\frac{9}{5}} x^{\frac{6}{5}} + b_4 F_{4,2}^{\frac{9}{5}} x^{\frac{1}{5}}, \tag{24}$$

where

$F_{2,0}^{\frac{9}{5}} = \frac{8}{\Gamma(\frac{6}{5})}, F_{3,0}^{\frac{9}{5}} = \frac{48}{\Gamma(\frac{11}{5})}, F_{3,1}^{\frac{9}{5}} = \frac{40}{\Gamma(\frac{6}{5})}, F_{4,0}^{\frac{9}{5}} = \frac{384}{\Gamma(\frac{16}{5})}, F_{4,0}^{\frac{9}{5}} = \frac{384}{\Gamma(\frac{16}{5})}, F_{4,1}^{\frac{9}{5}} = \frac{336}{\Gamma(\frac{11}{5})}$ and $F_{4,2}^{\frac{9}{5}} = \frac{120}{\Gamma(\frac{6}{5})}$.

The constants b_2, b_3 and b_4 are obtained by using (11) and then substituted back in (24) to get

$$D^{1.8}x^3 = \frac{5}{8}\frac{8}{\Gamma(\frac{6}{5})}x^{\frac{1}{5}} + \frac{1}{8}\frac{48}{\Gamma(\frac{11}{5})}x^{\frac{6}{5}} - \frac{1}{8}\frac{40}{\Gamma\left(\frac{6}{5}\right)}x^{\frac{1}{5}} = \frac{40}{\Gamma\left(\frac{11}{5}\right)}x^{\frac{6}{5}} \tag{25}$$

Since $b_2 = \frac{1}{4}$, $b_3 = \frac{1}{8}$ and $b_4 = 0$.

Methodology

Consider the fractional order wave equation given in (1), along with conditions (2) and (3). To implement Chebyshev collocation method, we shall let $u(x,t)$ be

$$u_n(x,t) = \sum_{j=0}^{n} u_j(t)W_j^*(x) \tag{26}$$

From (1), (21) and (22), we have

$$\sum_{j=0}^{n} \frac{d^2u_j(t)}{dt^2}W_j^*(x) = \sigma(x,t)\sum_{j-\lceil\alpha\rceil}^{n}\sum_{q=0}^{j-\lceil\alpha\rceil} u_j(t)F_{j,q}^{(\alpha)}x^{j-q-\alpha} + \rho(x,t) \tag{27}$$

Now, we shall collocate (27) at the points

$$x_p = (n + 1 - \lceil\alpha\rceil), \qquad p = 0,1,\dots\ ,n - \lceil\alpha\rceil.$$

as

$$\sum_{j=0}^{n} \ddot{u}_j(t)W_j^*(x_j) = \sigma(x_p,t)\sum_{j-\lceil\alpha\rceil}^{n}\sum_{q=0}^{j-\lceil\alpha\rceil} u_j(t)F_{j,q}^{(\alpha)}x^{j-q-\alpha} + \rho(x,t) \tag{28}$$

where $\ddot{u}(t)$ has its usual interpretation as $\frac{d^2u}{dt^2}$.

In order to have the best choice of collocation points, we use the roots of shifted Chebyshev polynomial equation $W_{n+1-\lceil\alpha\rceil}^* = 0$. Applying the initial condition on (26) and using (11), we obtain the constants u_j at the initial point $t = 0$. Also, applying the boundary condition on (26) on the interval $0 < x < 2$ to get $\lceil\alpha\rceil$ equations. That is,

$$\sum_{j=0}^{n}(-1)^j u_j(t) = 0, \qquad \sum_{j=0}^{n}(2j+1)u_j(t) = 0 \tag{29}$$

(28) together with $\lceil\alpha\rceil$ equations derived from the boundary conditions (29), give (n+1) system of ordinary differential equations which can be solved by using finite difference method to get the unknowns $u_j,\ j = 0,1,\dots,n$.

Numerical Experiment

In this section, the methods elucidated in the foregoing are put into use in solving space fractional order wave equation.

Problem

Consider the fractional order wave equation of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \sigma(x,t)\frac{\partial^{1.8}u(x,t)}{\partial x^{1.8}} + \rho(x,t) \tag{30}$$

Defined on a finite domain $0 < x < 2$ and $t > 0$ with coefficient functions

$$\sigma(x,t) = \Gamma(1.2)x^{1.8} \tag{31}$$

and the inhomogeneous source function

$$\rho(x,t) = 4e^{-t}x^2(2-x) - 16e^{-t}x^2 + 20e^{-t}x^3 \tag{32}$$

with initial conditions

$$u(x,0) = 4x^2(2-x), \qquad u_t(x,0) = -4x^2(2-x) \tag{33}$$

and Dirichlet boundary conditions

$$u(0,t) = u(2,t) = 0. \tag{34}$$

The exact solution to this problem is

$$u(x,t) = 4e^{-t}x^2(2-x). \tag{35}$$

Solution

We shall now consider an approximate solution of the form

$$u_n(x,t) = \sum_{j=0}^{n} u_j(t)W_j^*(x) \tag{36}$$

For $n = 3$, we have

$$u_3(x,t) = \sum_{j=0}^{3} u_j(t)W_j^*(x) \tag{37}$$

Using (37) in (28), we have

$$\sum_{j=0}^{3} \frac{d^2u_j(t)}{dt^2} W_j^*(x_p) = 1, \quad p = 0,1. \tag{38}$$

Using (28) and (29), we obtain the following system of ordinary differential equations

$$\left.\begin{array}{l} \ddot{u}_0(t) + g_1\ddot{u}_1(t) + g_2\ddot{u}_3(t) = R_1u_2(t) + R_2u_3(t) + \rho(x_0,t) \\ \ddot{u}_0(t) + g_{11}\ddot{u}_1(t) + g_{22}\ddot{u}_3(t) = R_{11}u_2(t) + R_{22}u_3(t) + \rho(x_1,t) \\ u_0(t) + u_1(t) + u_2(t) + u_3(t) = 0 \\ u_0(t) + 3u_1(t) + 5u_2(t) + 7u_3(t) = 0 \end{array}\right\}, \tag{39}$$

where

$g_1 = W_1^*(x_0)$, $g_2 = W_3^*(x_0)$, $g_{11} = W_1^*(x_1)$, $g_{22} = W_3^*(x_1)$, $R_1 = \sigma(x_0,t)F_{2.0}^{1.8}x_0^{\frac{1}{5}}$, $R_2 = \sigma(x_0,t)\left\{F_{3.0}^{1.8}x_0^{\frac{6}{5}} + F_{3.1}^{1.8}x_0^{\frac{1}{5}}\right\}$, $R_{11} = \sigma(x_1,t)F_{2.0}^{1.8}x_1^{\frac{1}{5}}$, $R_{22} = \sigma(x_1,t)\left\{F_{3.0}^{1.8}x_1^{\frac{6}{5}} + F_{3.1}^{1.8}x_1^{\frac{1}{5}}\right\}$.

In order to solve system of equations (36) using finite difference method, we will use the notations $t_i = i\Delta t$ for $i = 0, 1, \ldots, N$, with $\Delta t = \frac{T}{N}$ and $T = T_{final}$. Also, we defined

$$u_i^n = u_i(t_n), \qquad \rho_i^n = \rho_i(t_n) \tag{40}$$

Then, the system (39) becomes

$$\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{(\Delta t)^2} + g_1\frac{u_1^{n+1} - 2u_1^n + u_1^{n-1}}{(\Delta t)^2} + g_2\frac{u_3^{n+1} - 2u_3^n + u_3^{n-1}}{(\Delta t)^2} = R_1u_2^{n+1} + R_2u_3^{n+1} + \rho_0^{n+1} \tag{41}$$

$$\frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{(\Delta t)^2} + g_{11}\frac{u_1^{n+1} - 2u_1^n + u_1^{n-1}}{(\Delta t)^2} + g_{22}\frac{u_3^{n+1} - 2u_3^n + u_3^{n-1}}{(\Delta t)^2} = R_{11}u_2^{n+1} + R_{22}u_3^{n+1} + \rho_0^{n+1} \tag{42}$$

$$u_0^{n+1} - u_1^{n+1} + u_2^{n+1} - u_3^{n+1} = 0 \tag{43}$$

$$u_0^{n+1} + 3u_1^{n+1} + 5u_2^{n+1} + 7u_3^{n+1} = 0 \tag{44}$$

(41) - (44) can be written in matrix form as

$$\begin{pmatrix} 1 & g_1 & -(\Delta t)^2 R_1 & g_2-(\Delta t)^2 R_2 \\ 1 & g_{11} & -(\Delta t)^2 R_{11} & g_{22}-(\Delta t)^2 R_{22} \\ 1 & -1 & 1 & -1 \\ 1 & 3 & 5 & 7 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n+1} =$$

$$\begin{pmatrix} 2 & 2g_1 & 0 & 2g_2 \\ 2 & 2g_{11} & 0 & 2g_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n} - \begin{pmatrix} 1 & g_1 & 0 & g_2 \\ 1 & g_{11} & 0 & g_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{n-1}$$

$$+ (\Delta t)^2 \begin{pmatrix} \rho_0 \\ \rho_1 \\ 0 \\ 0 \end{pmatrix}^{n+1} \tag{45}$$

which is equivalent to

$$PU^{n+1} = QU^n - RU^{n-1} + (\Delta t)^2 S^{n+1} \tag{46}$$

$$U^{n+1} = P^{-1}QU^n - P^{-1}RU^{n-1} + P^{-1}(\Delta t)^2 S^{n+1}, \tag{47}$$

where

$U^n = (u_0^n, u_1^n, u_2^n, u_3^n)^T$ and $S^n = ((\Delta t)^2 \rho_0^n, (\Delta t)^2 \rho_1^n, 0,0)^T$.

To obtain the initial solution U^0 of (47), we use initial condition of the problem $u(x,0)$ combine with (11). We shall as well obtain U^1 of (47), we use the initial condition of the problem $u_t(x,0)$ as follows

$$u_t(x,0) = \frac{u_{i,1} - u_{i,0}}{\Delta t}, \tag{48}$$

Then

$$u_{i,1} = u_{i,0} + \Delta t u'(x_i), \tag{49}$$

hence

$$U^1 = U^0 + \Delta t G, \tag{50}$$

where

$$G = u(x_i). \tag{51}$$

The approximate solution is obtained by making substitution in the series (37).

RESULTS

Table 1: Absolute errors for the problem solved using the proposed method for

$$\Delta t = \frac{1}{400}, \ n = 3, \text{ and } T = 1.$$

x	Exact solution	Approximate solution	Error = |Exact- Approx.|
0.00	0	0	0
0.20	0.2872808993	0.2853666054	1.914294×10^{-3}
0.40	1.0188927790	0.9747779627	4.411482×10^{-2}
0.60	2.0009365580	1.9862317000	1.470486×10^{-2}
0.80	3.0414330890	3.1096670040	6.823392×10^{-2}
1.00	3.9503112020	4.0900055000	1.396943×10^{-1}
1.20	4.5395818000	4.6852897270	1.457079×10^{-1}
1.40	4.6223961170	4.6992014000	7.752402×10^{-2}
1.60	4.0148937660	3.9929906310	2.190314×10^{-2}
1.80	2.5343312070	2.4617225700	7.260864×10^{-2}
2.00	0	0	0

Table 2: Absolute errors for the problem solved using the proposed method for

$$\Delta t = \frac{1}{4000}, \ n = 3, \text{ and } T = 1.$$

x	Exact solution	Approximate solution	|Error = Exact- Approx |
0.00	0	0	0
0.20	0.2872808993	0.286354880	9.260113×10^{-4}
0.40	1.0188927790	1.0221455440	3.252765×10^{-3}
0.60	2.0009365580	2.0149684560	1.403190×10^{-2}
0.80	3.0414330890	3.0724231120	3.099002×10^{-2}
1.00	3.9503112020	4.0021090000	5.179780×10^{-2}
1.20	4.5395818000	4.6116256080	7.222980×10^{-1}
1.40	4.6223961170	4.7085724200	8.617630×10^{-2}
1.60	4.0148937660	4.1005489400	8.565517×10^{-2}
1.80	2.5343312070	2.5951546300	6.082342×10^{-2}
2.00	0	0	0

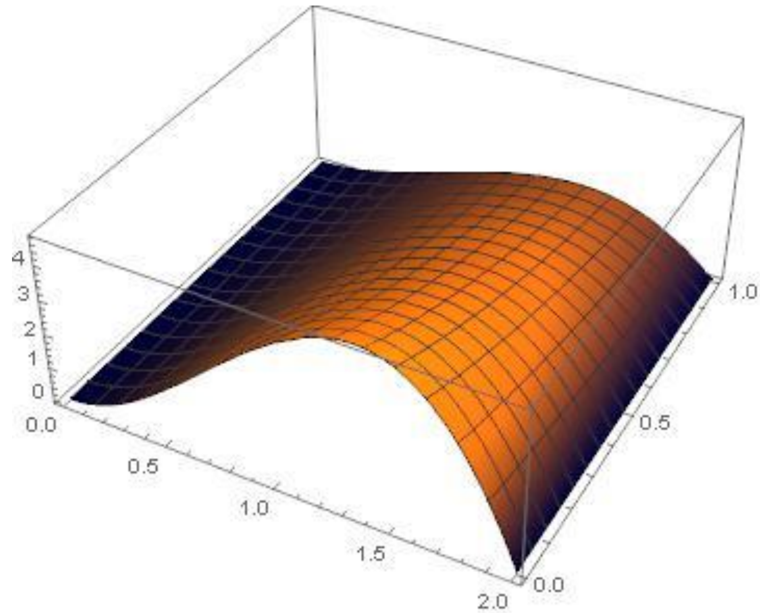

Figure 1: Exact solution graph

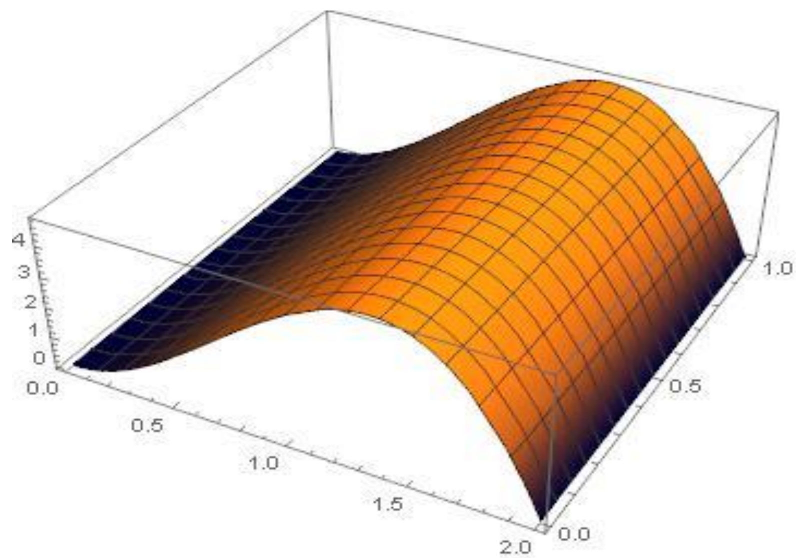

Figure 2: Approximate solution graph for $\Delta t = \dfrac{1}{400}$

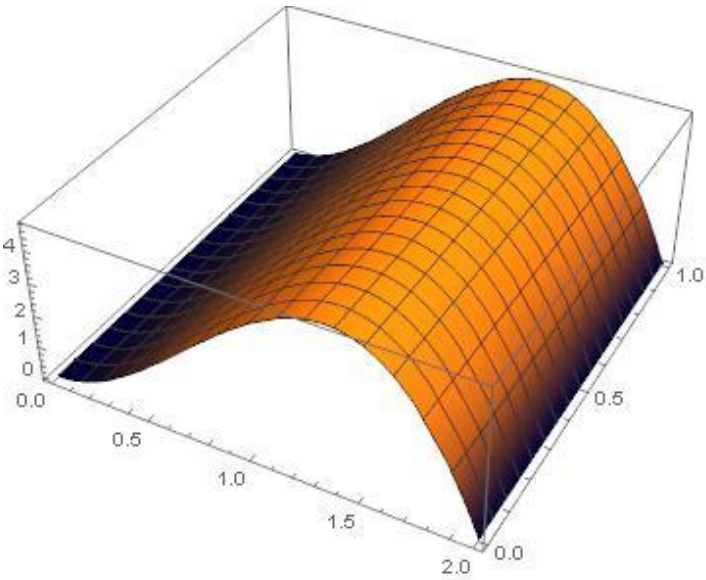

Figure 3: Approximate solution graph for $\Delta t = \dfrac{1}{4000}$

DISCUSSION

The tables 1 and 2 presented above depicts the relationship between the exact and the approximate solutions obtained using the proposed method for the case $n = 3$ at different step length $\Delta t = 0.0025$ and $\Delta t = 0.00025$ respectively. It is obvious that the result presented in column 4 of table 2 gives better approximation than that of column 4 of table 1. This observation is in line with the general understanding in numerical analysis that whenever a problem is not stiff, the smaller the step-size, the better the result. Other works in related approach are either not solving wave equation, that is, problems with the two independent variables being of second order, or when they do, the approach used are entirely different. For example, Crank-Nicolson scheme was implemented in (Sweilam & Nagy, 2011), a class of finite difference method was implemented for the solution of two-sided fractional wave equation in (Sweilam et al. 2011), etc. Meanwhile, their solutions could not be compared with ours for the obvious reasons of different approaches. The 3D graphs are for the exact solution, the approximate solution with the step-sizes $\frac{1}{400}$ and $\frac{1}{4000}$. It is observed that the accuracy of our solution is dependent of the step - size rather than the order of the given problem.

CONCLUSION

Numerical method of solving SFOWE has been proposed. The fractional derivative is interpreted in Caputo sense, with the given space fractional order PDE reduced to a system of second order ordinary differential equations. The system of second order ODEs thus obtained is reduced to a system of algebraic equations with the use of finite difference scheme and collocated at the roots of the shifted Chebyshev polynomials of the fourth kind. The approximate solutions obtained through the method proposed in this work are compared with the exact solution, and it is obvious that the proposed method is reliable. All computations are carried out using Mathematica.

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