FULL-LENGTH ARTICLE

Stability and Hopf Bifurcation Analysis of Maxwell-Bloch Equations

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Abstract
Stability theory plays a central role in system engineering, especially in the field of control systems and automation with regard to both dynamics and control. In this paper, stability and Hopf bifurcation analysis of Maxwell-Bloch equations were considered. By the aid of divergence test, it was proved that the system is dissipative. Steady state points of the equations were determined. The equations were linearized using Jacobian matrix about each equilibrium points. The local stability condition of each critical point was proved by using Routh- Huwertiz stability criteria. By the aid of Lyapunov theorem, equilibrium point one was proved to be globally asymptotically stable with some specific condition on pumping energy parameter. Furthermore, the result of Hopf bifurcation revealed that the system doesn't undergo Hopf bifurcation at equilibrium point one by any choice of pumping energy parameter and with some specific conditions the system undergoes Hopf bifurcation about the two remaining equilibrium points for a certain value of pumping energy parameter. Finally, in order to verify the applicability of the result two supportive examples were solved and MATLAB simulation was implemented to support the findings of the study.

Key words: Global stability; Hopf bifurcation; Local stability; Lyapunov theorem; Maxwell -Bloch equation; Routh- Huwertiz stability criteria.

INTRODUCTION

Maxwell-Bloch equations are set of coupled ordinary differential equations, which form the foundation of classical electromagnetism, classical optics and electric circuits together with the Lorenz force law. The equations also provide mathematical model for electric, optical and radio technologies, such as power generation, electric motors, wireless communication, etc (Maxwell, 1892).

The Maxwell-Bloch equations widely used in non-linear optics in general and to model quantum cascade lasers (QCL) (Jirauschek and Kubis, 2014). These model equations are a system of non-linear ordinary differential equations which plays a prominent role in the field of non-linear optics. Non-linear evolution equations have attracted a lot of attentions since they are able to describe the non-linear phenomena in many fields of sciences and Engineering (Ablowity and Clarkson, 2004).

Self-induced transparency (SIT) phenomenon plays a role in overcoming the attenuation in the optical communication systems.Therefore, scholars or researchers have pointed out the reduced Maxwell-Bloch equations can be applied to get for the phenomenon of self-induced transparency (We and Zhang, 2016). In general, mathematical models of Maxwell-Bloch equations are used in Physics, Chemistry, Biology, Engineering disciplines and others related sciences. In 1965, Tito Arecchi and Rodolfo Bonifacio of Milan discovered the

Maxwell-Bloch equations which are a system of non-linear ordinary differential equations of the form:

$$\begin{aligned} \frac{dx}{dt} &= k\left(y-x\right) \\ \frac{dy}{dt} &= r_1\left(xz-y\right) \\ \frac{dz}{dt} &= r_2\left(\lambda+1-z-\lambda xy\right) \end{aligned} \quad , \qquad (1.1)$$

where the parameter λ may be positive, negative or zero, k, r_1 and r_2 are positive parameters. λ is a pumping energy parameter, k is the decay rate in the laser cavity due to beam transmission, r_1 is the decay rate of the atomic polarization, r_2 is the decay rate of the population inversion, x is the dynamics of the electric field, y is Atomic polarization and z is the population inversion.

Non-linear mathematical models of real-world phenomena that are formulated in terms of ordinary differential equations as in Eq. (1.1) are not easy to directly solve for their solution and hence it is necessary to use qualitative approaches, such as stability and bifurcation analysis, to investigate their solution behaviors. In scientific fields as diverse as fluid mechanics, electronics, chemistry and theoretical ecology, there is the application of what is referred to as bifurcation analysis; the analysis of a system of non-linear ordinary differential equations under parameter variation (Blanchard *et al.*, 2006).

Hopf bifurcation is a local bifurcation in which a fixed point of a dynamical system loses stability as a pair of complex eigenvalues of linearized system crosses the imaginary axis of the complex plane. Hassard *et al.* (1981) studied the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions by applying the normal form theory and the center manifold theorem. Nijamuddin and Santabrata (2015) proposed the stability and bifurcation analysis of three species competitive food chain model system incorporating prey-refuge and this study showed that competition among predators could be beneficial for predators. Tee and Salleh (2016) investigated Hopf bifurcation of non-linear modified Lorenz system using normal form theory that was the technique used in Hassard *et al.*(1981). Algaba *et al*.(2016) studied the local bifurcations of equilibrium in the Lorenz system, when the parameters are allowed to take any real value has been successfully completed in the case of the pitchfork and Hopf bifurcations. Furthermore, Yang *et al*. (2017) described chamostat model which involve control strategy with threshold window are analyzed. They investigated the qualitative analysis such as existence and stability of equilibrium points of the system and proved that pseudo-equilibrium cannot coexist.

Recently, Pijush *et al*.(2018) investigated the stability and bifurcation analysis of three-species food chain model with fear and they concluded that chaotic dynamics can be controlled by the fear factors. Most recently, Makwata *et al*.(2019) investigated stability and bifurcation analysis of fishery model with allee effects and they obtained the three different equilibrium solutions as one being stable and with two being saddles.

However, there is paucity of information with regard to Stability and Hopf Bifurcation analysis of Maxwell-Bloch equations in the existing literature. Therefore, the main objective of this paper is stability and Hopf Bifurcation analysis of Maxwell-Bloch equations given by equation (1.1)

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RESULT

Dissipative or Conservative of the System

Consider system (1.1) given by

$$\frac{dx}{dt} = k\left(y - x\right)$$
$$\frac{dy}{dt} = r_1\left(xz - y\right)$$
$$\frac{dz}{dt} = r_2\left(\lambda + 1 - z - \lambda xy\right)$$

Let

$$\begin{aligned} f_1 &= k(y-x) \\ f_2 &= r_1(xz-y) \\ f_3 &= r_2(\lambda+1-z-\lambda xy) \end{aligned} \tag{2.1}$$

From the system (2.1),

$$\frac{\partial f_1\left(x,y,z\right)}{\partial x} = -k \;, \quad \frac{\partial f_2\left(x,y,z\right)}{\partial y} = -r_1 \;, \quad \frac{\partial f_3\left(x,y,z\right)}{\partial z} = -r_2$$

$$\nabla . f\left(x,y,z\right) = \frac{\partial f_1\left(x,y,z\right)}{\partial x} + \frac{\partial f_2\left(x,y,z\right)}{\partial y} + \frac{\partial f_3\left(x,y,z\right)}{\partial z}$$
$$= -\left(k + r_1 + r_2\right)$$

$$\frac{dD}{dt} = \int \nabla . f dD = \int -(k + r_1 + r_2) dD$$

$$\frac{dD}{dt} = -(k + r_1 + r_2) D$$

$$\frac{1}{D} dD = -\left(k + r_1 + r_2\right) dt$$

$$\int \frac{1}{D} dD = -\int \left(k + r_1 + r_2\right) dt$$

$$D = D_0 e^{-\left(k + r_1 + r_2\right)t}$$

D is decreasing exponentially

Therefore, the system (1.1) is dissipative.

Equilibrium Points of the System

To find the equilibrium point, equate the system (1.1) with zero and upon simplification the following three equilibrium points were obtained.

$$E_1 = \left(0,0,\lambda+1\right), E_2 = \left(-1,-1,1\right) \text{ and } E_3 = \left(1,1,1\right)$$

Local Stability Analysis

Linearizing system (1.1) at each equilibrium points and state the local stability conditions of the system. The Jacobian matrix of the system (1.1) is:

$$A = \begin{pmatrix} -k & k & 0 \\ r_1 z & -r_1 & r_1 x \\ -r_2 \lambda y & -r_2 \lambda x & -r_2 \end{pmatrix}$$

The Jacobian matrix evaluated at the first equilibrium point

$E_1 = (0,0,\lambda+1)$ is $J = A\big|E_1 = (0,0,\lambda+1)$

$$J = \begin{pmatrix} -k & k & 0 \\ r_1(\lambda+1) & -r_1 & 0 \\ 0 & 0 & -r_2 \end{pmatrix} \tag{2.2}$$

The characteristic equation for equation (2.2) is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -k-m & k & 0 \\ r_1(\lambda+1) & -r_1 - m & 0 \\ 0 & 0 & -r_2 - m \end{vmatrix} = 0$$

$$m^3 + (k + r_1 + r_2)m^2 + (kr_2 + r_1 r_2 - kr_1\lambda)m - kr_1 r_2 \lambda = 0$$

$$m^3 + a_1 m^2 + a_2 m + a_3 = 0 \quad , \tag{2.3}$$

where

$$\begin{aligned} a_1 &= k + r_1 + r_2 \\ a_2 &= kr_2 + r_1 r_2 - kr_1\lambda \\ a_3 &= -kr_1 r_2 \lambda \end{aligned} \tag{2.4}$$

Applying Routh-Hurwitz stability criterion for characteristic equation (2.3)

$1 > 0$

$a_1 = k + r_1 + r_2 > 0$, since, k, r_1, r_2 are positive parameters

$a_1 a_2 - a_3 = kr_2(k + 2r_1 + r_2) + r_1 r_2 (r_1 + r_2) - k\lambda r_1 (k + r_1) > 0$

$a_3 = -kr_1 r_2 \lambda > 0$

If

$$\lambda < 0 \tag{2.5}$$

As a result, the system (1.1) is locally asymptotically stable at the equilibrium point

$E_1 = (0, 0, \lambda + 1)$ provided that condition (2.5) is satisfied.

The Jacobian matrix evaluated at the second equilibrium point $E_2 = (-1, -1, 1)$ is

$$J = A|E_2 = (-1, -1, 1)$$

$$J = \begin{pmatrix} -k & k & 0 \\ r_1 & -r_1 & -r_1 \\ r_2\lambda & r_2\lambda & -r_2 \end{pmatrix} \tag{2.6}$$

The characteristic equation for equation (2.6) is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -k - m & k & 0 \\ r_1 & -r_1 - m & -r_1 \\ r_2\lambda & r_2\lambda & -r_2 - m \end{vmatrix} = 0$$

$$m^3 + (k + r_1 + r_2)m^2 + (kr_2 + r_1 r_2 + r_1 r_2 \lambda)m + 2kr_1 r_2 \lambda = 0$$

$$m^3 + b_1 m^2 + b_2 m + b_3 = 0, \tag{2.7}$$

where

$$\begin{aligned} b_1 &= k + r_1 + r_2 \\ b_2 &= kr_2 + r_1 r_2 + r_1 r_2 \lambda \\ b_3 &= 2kr_1 r_2 \lambda \end{aligned} \tag{2.8}$$

The Routh array or Routh-Hurwitz table is

$$\begin{array}{c|ccc} m^3 & 1 & b_2 & 0 \\ m^2 & b_1 & b_3 & 0 \\ m^1 & \dfrac{b_1 b_2 - b_3}{b_1} & 0 & \\ m^0 & b_3 & & \end{array}$$

Applying Routh-Hurwitz stability criterion for characteristic equation (2.7)

$1 > 0$

$b_1 = k + r_1 + r_2 > 0$, since, k, r_1, r_2 are positive parameters

$$b_1 b_2 - b_3 = kr_2(k + 2r_1 + r_2) + r_1 r_2 (r_1 + r_2 + \lambda r_1 + \lambda r_2) - kr_1 r_2 \lambda > 0$$

$$b_3 = 2kr_1 r_2 \lambda > 0$$

If

$$\lambda > 0 \tag{2.9}$$

Therefore, the system (1.1) is locally asymptotically stable at the equilibrium point $E_2 = (-1,-1,1)$ provided that condition (2.9) is satisfied.

The Jacobian matrix evaluated at the third equilibrium point $E_3 = (1,1,1)$ is $J = A|E_3 = (1,1,1)$

$$J = \begin{pmatrix} -k & k & 0 \\ r_1 & -r_1 & r_1 \\ -r_2\lambda & -r_2\lambda & -r_2 \end{pmatrix}$$

The characteristic equation for equation (2.10) is $|J - mI| = 0$

$$|J - mI| = \begin{vmatrix} -k-m & k & 0 \\ r_1 & -r_1 - m & r_1 \\ -r_2\lambda & -r_2\lambda & -r_2 - m \end{vmatrix} = 0$$

$$m^3 + (k + r_1 + r_2)m^2 + (kr_2 + r_1r_2 + r_1r_2\lambda)m + 2kr_1r_2\lambda = 0$$

$$m^3 + c_1m^2 + c_2m + c_3 = 0, \tag{2.11}$$

where

$$\begin{aligned} c_1 &= k + r_1 + r_2 \\ c_2 &= kr_2 + r_1r_2 + r_1r_2\lambda \\ c_3 &= 2kr_1r_2\lambda \end{aligned} \tag{2.12}$$

The Routh array or Routh-Hurwitz table is:

$$\begin{array}{c|ccc} m^3 & 1 & c_2 & 0 \\ m^2 & c_1 & c_3 & 0 \\ m^1 & \dfrac{c_1c_2 - c_3}{c_1} & 0 & \\ m^0 & c_3 & & \end{array}$$

Applying Routh-Hurwitz stability criterion for characteristic equation (2.11)

$1 > 0$

$c_1 = k + r_1 + r_2 > 0$, since, k, r_1, r_2 are positive parameters

$c_1c_2 - c_3 = kr_2(k + 2r_1 + r_2) + r_1r_2(r_1 + r_2 + \lambda r_1 + \lambda r_2) - kr_1r_2\lambda > 0$

$$c_3 = 2kr_1r_2\lambda > 0 \text{ if } \lambda > 0 \tag{2.13}$$

As a result, the system (1.1) is locally asymptotically stable at the equilibrium point $E_3=(1,1,1)$ provided that condition (2.13) is satisfied.

Global Stability Analysis of the System

To analyze the global asymptotic stability of non-linear system (1.1)

Let $v_1(x,y,z)=\frac{1}{k}x^2+\frac{1}{r_1}y^2+\frac{1}{r_2}(z-\lambda-1)^2$ be candidate Lyapunov function at equilibrium point

$E_1=(0,0,\lambda+1)$, then:-

1. $v_1(x^*,y^*,z^*)=v_1(0,0,\lambda+1)=0$
2. $v_1(x,y,z)$ >0 for all $(x,y,z)\in D-\{(x^*,y^*,z^*)\}$

   Hence, $v_1(x,y,z)$ is positive definite function.
3. $\frac{dv_1}{dt}(x,y,z)=\frac{\partial v_1}{\partial x}(x,y,z)\frac{dx}{dt}+\frac{\partial v_1}{\partial y}(x,y,z)\frac{dy}{dt}+\frac{\partial v_1}{\partial z}(x,y,z)\frac{dz}{dt}$

   $\frac{dv_1}{dt}(x,y,z)=-2g(x,y,z)$

where

$$g(x,y,z)=x^2+y^2+z^2+\lambda^2-\lambda^2xy+\lambda xyz-xyz-\lambda xy-xy-2\lambda z-2z+2\lambda+1$$

Construct Hessian matrix for $g(x,y,z)$ at the first equilibrium point $E_1=(0,0,\lambda+1)$ to check
whether Hessian matrix is positive definite.

$$H=\left.\begin{pmatrix}2 & -\lambda-2 & 0\\ -\lambda-2 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}\right|_{E_1=(0,0,\lambda+1)}$$

All leading principal minors of Hessian matrix at $E_1=(0,0,\lambda+1)$ are:

$D_1=2$, $D_2=\begin{vmatrix}2 & -\lambda-2\\ -\lambda-2 & 2\end{vmatrix}=-\lambda(\lambda+4)$

and $D_3=\begin{vmatrix}2 & -\lambda-2 & 0\\ -\lambda-2 & 2 & 0\\ 0 & 0 & 2\end{vmatrix}=-2\lambda(\lambda+4)$

The leading principal minors are:

$D_1=2>0,$

$D_2=-\lambda(\lambda+4)>0,$

$D_3=-2\lambda(\lambda+4)>0$ If $\lambda\in(-4,0)$ (2.14)

The Hessian matrix is positive definite if condition (2.14) is satisfied.

So that $\frac{dv_1}{dt}(x,y,z)$ is negative definite function when condition (2.14) is satisfied.

4. $\lim_{(x,y,z)\to\infty} v_1(x,y,z) = \lim_{(x,y,z)\to\infty}\left[\frac{x^2}{k}+\frac{y^2}{r_1}+\frac{1}{r_2}(z-\lambda-1)^2\right]=\infty$

$v_1(x,y,z)$ is radially unbounded.

As a result, the equilibrium point $E_1=(0,0,\lambda+1)$ is globally asymptotically stable by Lyapunov Stability theorem if condition (2.14) is satisfied.

Hopf Bifurcation Analysis of the System

Suppose that the system (1.1) has critical point for some parameter $\lambda=\lambda_0$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real part. Furthermore, Let $\mathrm{Re}\left(\frac{d\lambda}{dm}\right)\neq 0$, then the Hopf bifurcation occurs at $\lambda=\lambda_0$.

Let the characteristic equation (2.3) has pure imaginary eigenvalues $m=\pm i\omega(\omega>0)$ at $\lambda=\lambda_0$

$$m^3+a_1m^2+a_2m+a_3=0$$
$$(\omega i)^3+a_1(\omega i)^2+a_2(\omega i)+a_3=0$$
$$(-\omega^3+a_2\omega)i-a_1\omega^2+a_3=0$$

Equating the real and imaginary parts with zeros yields.

$$a_1a_2-a_3=0 \tag{2.15}$$

Substituting equation (2.4) into equation (2.15) to compute for parameter λ yield

$$(k+r_1+r_2)(kr_2+r_1r_2-kr_1\lambda)+kr_1r_2\lambda=0$$
$$\lambda=\frac{r_2(k+r_1+r_2)}{kr_1} \tag{2.16}$$

Plugging equation (2.16) into equation (2.4)

$$\begin{aligned} a_1 &= k+r_1+r_2 \\ a_2 &= -r_2^2 \\ a_3 &= -kr_1r_2\left(\frac{r_2(k+r_1+r_2)}{kr_1}\right) \\ &= -r_2^2(k+r_1+r_2) \end{aligned} \tag{2.17}$$

Substituting equation (2.17) into equation (2.3) yields.

$$m_1 = -(k + r_1 + r_2) \text{ or } m^2 = r_2^{\,2}$$
$$m_{2,3} = \pm r_2$$

Since $m_{2,3}$ are not pure imaginary eigenvalues, then one of Hopf bifurcation condition is not satisfied.

As a result, the system (1.1) does not undergo Hopf bifurcation at equilibrium one by any choice of pumping energy parameter.

Let the characteristic equation (2.7) has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$

$$m^3 + b_1 m^2 + b_2 m + b_3 = 0$$
$$(i\omega)^3 + b_1(i\omega)^2 + b_2(i\omega) + b_3 = 0$$
$$(-\omega^3 + b_2\omega)i - b_1\omega^2 + b_3 = 0$$

Equating the real and imaginary parts with zeros yields.

$$b_1 b_2 - b_3 = 0 \tag{2.18}$$

Plugging equation (2.8) into equation (2.18) to calculate for λ

$$(k + r_1 + r_2)(kr_2 + r_1 r_2 + r_1 r_2 \lambda) - 2kr_1 r_2 \lambda = 0$$

$$\lambda = \frac{-(k + r_1)(k + r_1 + r_2)}{r_1(r_1 + r_2 - k)} \tag{2.19}$$

Substituting equation (2.19) into equation (2.8)

$$b_1 = k + r_1 + r_2$$
$$b_2 = \frac{-2kr_2(k + r_1)}{r_1 + r_2 - k} \tag{2.20}$$
$$b_3 = \frac{-2kr_2(k + r_1)(k + r_1 + r_2)}{r_1 + r_2 - k}$$

$$\omega^2 = \frac{-2kr_2(k + r_1)}{r_1 + r_2 - k}$$

$$\omega = \pm\sqrt{\frac{2kr_2(k + r_1)}{k - (r_1 + r_2)}} \text{ if } k > r_1 + r_2 \tag{2.21}$$

Substituting equation (2.20) into equation (2.7)

$$m_1 = -(k + r_1 + r_2) \text{ or } m^2 = \frac{2kr_2(k + r_1)}{r_1 + r_2 - k}$$

$$m_{2,3} = \pm\sqrt{\frac{2kr_2(k + r_1)}{r_1 + r_2 - k}}$$

Since $k > r_1 + r_2$

$$m_{2,3} = \pm\sqrt{\frac{-2kr_2(k + r_1)}{k - (r_1 + r_2)}}$$

$$= \pm i\sqrt{\frac{2kr_2(k + r_1)}{k - (r_1 + r_2)}}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied if condition (2.21) is satisfied.

Next compute the $\frac{dm}{d\lambda}$ from the characteristic equation of the Jacobian matrix for equation (2.7)

$$\frac{dm}{d\lambda} = -\left(\frac{r_1 r_2 m + 2kr_1 r_2}{3m^2 + 2b_1 m + b_2}\right)$$

$$\frac{d\lambda}{dm} = \left(\frac{dm}{d\lambda}\right)^{-1} = -\left(\frac{3m^2 + 2b_1 m + b_2}{2kr_1 r_2 + r_1 r_2 m}\right)$$

$$= -\left(\frac{3(i\omega)^2 + 2b_1(i\omega) + b_2}{2kr_1 r_2 + r_1 r_2(i\omega)}\right)$$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{2(3k\omega^2 - b_2 k - b_1\omega^2)}{r_1 r_2(4k^2 + \omega^2)} = \frac{b_2(4k - 2b_1)}{r_1 r_2(4k^2 + b_2)} \neq 0$$

Since $\operatorname{Re}\left(\frac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied if condition (2.21) is satisfied. As a result, the system (1.1) under goes Hopf bifurcation at $\lambda = \frac{-(k + r_1)(k + r_1 + r_2)}{r_1(r_1 + r_2 - k)}$ when condition (2.21) is satisfied.

Suppose the characteristic equation (2.11) has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at

$\lambda = \lambda_0$

$$m^3 + c_1 m^2 + c_2 m + c_3 = 0$$

$$(i\omega)^3 + c_1(i\omega)^2 + c_2(i\omega) + c_3 = 0$$
$$\left(-\omega^3 + c_2\omega\right)i - c_1\omega^2 + c_3 = 0$$

Equating the real and imaginary parts with zeros yields.

$$c_1c_2 - c_3 = 0 \tag{2.22}$$

Plugging equation (2.12) into equation (2.22) to compute for λ

$$(k + r_1 + r_2)(kr_2 + r_1r_2 + r_1r_2\lambda) - 2kr_1r_2\lambda = 0$$

$$\lambda = \frac{-(k + r_1)(k + r_1 + r_2)}{r_1(r_1 + r_2 - k)} \tag{2.23}$$

Substituting equation (2.23) into equation (2.12)

$$c_1 = k + r_1 + r_2$$
$$c_2 = \frac{-2kr_2(k + r_1)}{r_1 + r_2 - k} \tag{2.24}$$
$$c_3 = \frac{-2kr_2(k + r_1)(k + r_1 + r_2)}{r_1 + r_2 - k}$$

$$\omega^2 = \frac{-2kr_2(k + r_1)}{r_1 + r_2 - k}$$

$$\omega = \pm\sqrt{\frac{2kr_2(k + r_1)}{k - (r_1 + r_2)}} \text{ if } k > r_1 + r_2 \tag{2.25}$$

Substituting equation (2.24) into equation (2.11)

$$m_1 = -(k + r_1 + r_2) \text{ or } m^2 = \frac{2kr_2(k + r_1)}{r_1 + r_2 - k}$$

$$m_{2,3} = \pm\sqrt{\frac{2kr_2(k + r_1)}{r_1 + r_2 - k}}$$

Since $k > r_1 + r_2$

$$m_{2,3} = \pm\sqrt{\frac{-2kr_2(k + r_1)}{k - (r_1 + r_2)}}$$
$$= \pm i\sqrt{\frac{2kr_2(k + r_1)}{k - (r_1 + r_2)}}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied if condition (2.25) is satisfied. Next compute the $\frac{dm}{d\lambda}$ from the characteristic equation of the Jacobian matrix for equation (2.11)

$$\frac{dm}{d\lambda} = -\left(\frac{r_1 r_2 m + 2kr_1 r_2}{3m^2 + 2c_1 m + c_2}\right)$$

$$\frac{d\lambda}{dm} = \left(\frac{dm}{d\lambda}\right)^{-1} = -\left(\frac{3m^2 + 2c_1 m + c_2}{2kr_1 r_2 + r_1 r_2 m}\right)$$

$$= -\left(\frac{3(i\omega)^2 + 2c_1(i\omega) + c_2}{2kr_1 r_2 + r_1 r_2 (i\omega)}\right)$$

$$= \frac{3\omega^2 - c_2 - 2c_1\omega i}{2kr_1 r_2 + r_1 r_2 \omega i}\left(\frac{2kr_1 r_2 - r_1 r_2 \omega i}{2kr_1 r_2 - r_1 r_2 \omega i}\right)$$

$$= \frac{2\left(3k\omega^2 - c_2 k - c_1\omega^2\right)}{r_1 r_2\left(4k^2 + \omega^2\right)} - \frac{\left(3\omega^2 - c_2 + 4c_1 k\right)\omega i}{r_1 r_2\left(4k^2 + \omega^2\right)}$$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{2\left(3k\omega^2 - c_2 k - c_1\omega^2\right)}{r_1 r_2\left(4k^2 + \omega^2\right)} = \frac{c_2\left(4k - 2c_1\right)}{r_1 r_2\left(4k^2 + c_2\right)} \neq 0$$

Since $\operatorname{Re}\left(\frac{d\lambda}{dm}\right) \neq 0$, then it proves that the second condition of Hopf bifurcation is satisfied if condition (2.25) is satisfied.

Therefore, the system (1.1) undergoes Hopf bifurcation at $\lambda = \frac{-(k + r_1)(k + r_1 + r_2)}{r_1(r_1 + r_2 - k)}$ when condition (2.25) is satisfied.

Supportive Examples

Example 1. Consider the parameters values with $r_1 = r_2 = 1,\ k = 4,\ \lambda = 2$, then the system (1.1) becomes

$$\frac{dx}{dt} = 4(y - x)$$

$$\frac{dy}{dt} = (xz - y)$$

$$\frac{dz}{dt} = (3 - z - 2xy)$$

Dissipative or Conservative of the System

Let $f_1 = 4(y-x)$
$f_2 = (xz-y)$
$f_3 = (3-z-\lambda xy)$

$$\frac{\partial f_1}{\partial x} = -4\ , \qquad \frac{\partial f_2}{\partial y} = -1\ , \qquad \frac{\partial f_3}{\partial z} = -1$$

The divergence of the vector field $\nabla . f\left(x,y,z\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(k + r_1 + r_2) = -6$

$$D = D_0 e^{-6t}$$

D is decreasing exponentially
Therefore, the system is dissipative.
Equilibrium points of the system,

Three equilibrium points of the system are as follows:-

$$E_1 = \left(0,0,\lambda+1\right) = \left(0,0,3\right)$$
$$E_2 = \left(-1,-1,1\right) = \left(-1,-1,1\right)$$
$$E_3 = \left(1,1,1\right) = \left(1,1,1\right)$$

Local Stability Analysis

The Jacobian matrix evaluated at the first equilibrium point

$E_1 = \left(0,0,3\right)$ is $J = A\big|E_1 = \left(0,0,3\right)$

$$J = \begin{pmatrix} -4 & 4 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at $E_1 = \left(0,0,3\right)$ is $\left|J - mI\right| = 0$

$m^3 + 6m^2 - 3m - 8 = 0$ where $a_1 = 6,\quad a_2 = -3,\ a_3 = -8$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at

$E_1 = \left(0,0,3\right)$ equilibrium point one is unstable. The Jacobian matrix evaluated at the second equilibrium point

$E_2 = \left(-1,-1,1\right)$ is $J = A\big|E_2 = \left(-1,-1,1\right)$

$$J = \begin{pmatrix} -4 & 4 & 0 \\ 1 & -1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at $E_2 = \left(-1,-1,1\right)$ is $\left|J - mI\right| = 0$

$m^3 + 6m^2 + 7m + 16 = 0$ where $b_1 = 6,\quad b_2 = 7,\ b_3 = 16$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$, equilibrium point two is locally asymptotically stable.

The Jacobian matrix evaluated at the third equilibrium point $E_3 = (1,1,1)$ is

$J = A|E_3 = (1,1,1)$ $$J = \begin{pmatrix} -4 & 4 & 0 \\ 1 & -1 & 1 \\ -2 & -2 & -1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$ is $|J - mI| = 0$

$$m^3 + 6m^2 + 7m + 16 = 0 \text{ where } c_1 = 6,\ c_2 = 7,\ c_3 = 16$$

Applying Routh-Hurwitz stability criterion for characteristic equation of the Jacobian matrix at

$E_3 = (1,1,1)$, equilibrium point three is locally asymptotically stable.

MATLAB Simulation

The following diagrams indicate MATLAB simulation that shows stability of the equilibrium point.

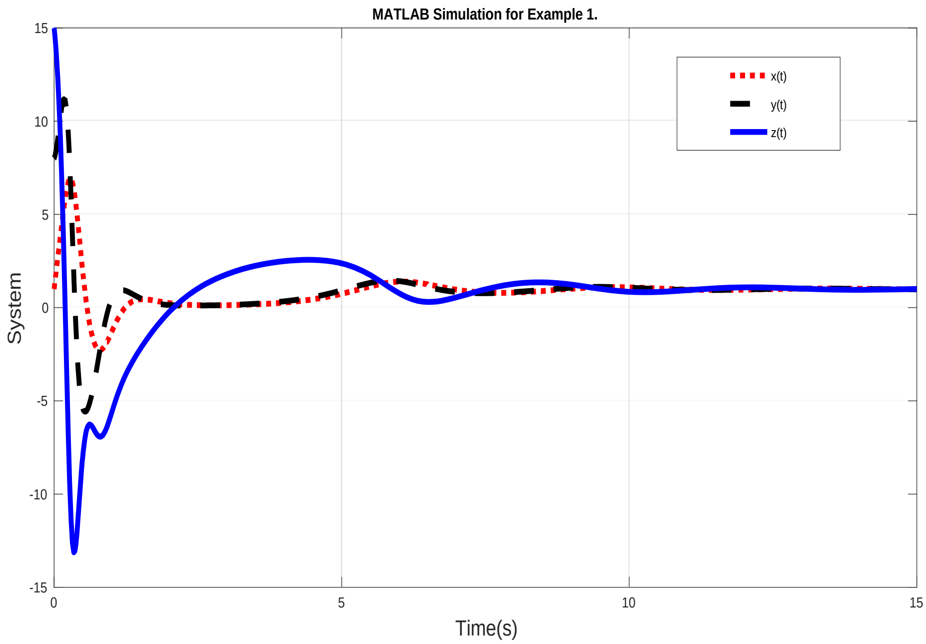

Figure 1: The graph of system versus time about equilibrium point.

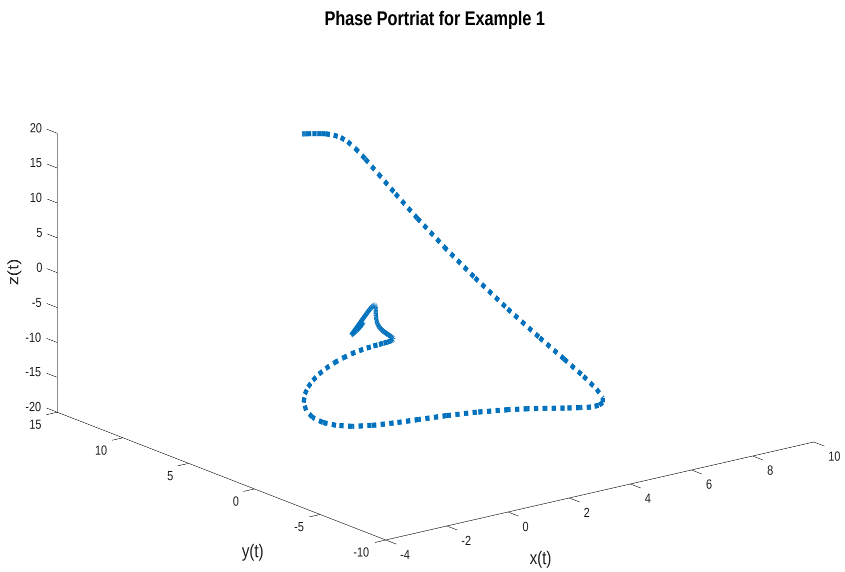

Figure 2: Phase Portrait of the system about the equilibrium point.

DISCUSSION

Figure 1 indicates that the graph of the system versus time converges to the equilibrium point. Figure 2 indicates phase portrait of the system in which the trajectory of the system converges to equilibrium point which in one ways shows the stability of the equilibrium point.

Hopf Bifurcation Analysis of the System,

Suppose the characteristic equation at $E_1 = (0,0,3)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$, then $m^3 + a_1m^2 + a_2m + a_3 = 0$

Where $\lambda = \dfrac{r_2(k + r_1 + r_2)}{kr_1} = \dfrac{3}{2}$

$$a_1 = k + r_1 + r_2 = 6$$

$$a_2 = -r_2^{\,2} = -1$$

$$a_3 = -r_2^{\,2}(k + r_1 + r_2) = -6$$

$$m^3 + 6m^2 - m - 6 = 0$$

$$(m+6)(m^2 - 1) = 0$$

$$m_1 = -6 \text{ or } m_{2,3} = \pm 1$$

Since $m_{2,3}$ are not pure imaginary eigenvalues, then one of Hopf bifurcation condition is not satisfied. As a result, the system does not undergo Hopf bifurcation at $\lambda = \frac{3}{2}$.

Suppose the characteristic equation of the Jacobian matrix at $E_2 = (-1,-1,1)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$, then $m^3 + b_1 m^2 + b_2 m + b_3 = 0$

Where $\lambda = \dfrac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)} = 15$

$$b_1 = k + r_1 + r_2 = 6$$

$$b_2 = \frac{-2kr_2(k+r_1)}{r_1+r_2-k} = 20$$

$$b_3 = \frac{-2kr_2(k+r_1)(k+r_1+r_2)}{r_1+r_2-k} = 120$$

$$m^3 + 6m^2 + 20m + 120 = 0$$

$$(m+6)(m^2+20) = 0$$

$$m_1 = -6\ , m_{2,3} = \pm i\sqrt{20}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then one of Hopf bifurcation condition is satisfied at $\lambda = 15$.

Next compute the $\operatorname{Re}\left(\frac{d\lambda}{dm}\right)$ from the characteristic equation of the Jacobian matrix at

$$E_2 = (-1,-1,1)$$

$$\operatorname{Re}\left(\frac{d\lambda}{dm}\right) = \frac{b_2(4k-2b_1)}{r_1 r_2(4k^2+b_2)} = \frac{20}{21} \neq 0$$

Since $\operatorname{Re}\left(\frac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied .

As a result, the system under goes Hopf bifurcation at $\lambda = 15$.

Let the characteristic equation of the Jacobian matrix at $E_3 = (1,1,1)$ has pure imaginary eigenvalues $m = \pm i\omega(\omega > 0)$ at $\lambda = \lambda_0$, then $m^3 + c_1 m^2 + c_2 m + c_3 = 0$

Where $\lambda = \dfrac{-(k+r_1)(k+r_1+r_2)}{r_1(r_1+r_2-k)} = 15$

$$c_1 = k + r_1 + r_2 = 6$$

$$c_2 = \frac{-2kr_2\left(k + r_1\right)}{r_1 + r_2 - k} = 20$$

$$c_3 = \frac{-2kr_2\left(k + r_1\right)\left(k + r_1 + r_2\right)}{r_1 + r_2 - k} = 120$$

$$m^3 + 6m^2 + 20m + 120 = 0$$

$$\left(m + 6\right)\left(m^2 + 20\right) = 0$$

$$m_1 = -6\ , m_{2,3} = \pm i\sqrt{20}$$

Since $m_{2,3}$ are pure imaginary eigenvalues, then the condition one of the Hopf bifurcation is
satisfied at $\lambda = 15$.
Next compute the $\mathrm{Re}\left(\frac{d\lambda}{dm}\right)$ from the characteristic equation of the Jacobian matrix at

$E_3 = \left(1,1,1\right)$

$$\mathrm{Re}\left(\frac{d\lambda}{dm}\right) = \frac{c_2\left(4k - 2c_1\right)}{r_1 r_2\left(4k^2 + c_2\right)} = \frac{20}{21} \neq 0$$

Since $\mathrm{Re}\left(\frac{d\lambda}{dm}\right) \neq 0$, then second condition of Hopf bifurcation is satisfied .

Therefore, the system undergoes Hopf bifurcation at $\lambda = 15$.

Example 2. Consider parameters values with $k = 0.4,$ $r_1 = 0.2$, $r_2 = 0.1,$ $\lambda = -0.1$, then the system (1.1) becomes

$$\frac{dx}{dt} = 0.4(y - x)$$

$$\frac{dy}{dt} = 0.2(xz - y)$$

$$\frac{dz}{dt} = 0.1(0.9 - z + 0.1xy)$$

Dissipative or Conservative of the System

$$D = D_0 e^{-0.7t} \quad \left(D_0 = e^c\right)$$

D is decreasing exponentially
Therefore, the system is dissipative.
Equilibrium points of the system,

We end up with three equilibrium points of the system as follows:-

$$E_1 = \left(0, 0, \lambda + 1\right) = \left(0, 0, 0.9\right)$$

$$E_2 = \left(-1, -1, 1\right) = \left(-1, -1, 1\right)$$

$$E_3 = \left(1,1,1\right) = \left(1,1,1\right)$$

Local Stability Analysis

The Jacobian matrix evaluated at the first equilibrium point

$E_1 = (0,0,0.9)$ is $J = A|E_1 = (0,0,0.9)$

$$J = \begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.18 & -0.2 & 0 \\ 0 & 0 & -0.1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at equilibrium point E_1 is $|J - mI| = 0$

$$m^3 + 0.7m^2 + 0.068m + 0.0008 = 0 \text{ where } a_1 = 0.7, \quad a_2 = 0.068, \; a_3 = 0.0008$$

Since there is no sign changes in the first column of Routh- Hurwitz table,
equilibrium point one is locally asymptotically stable.
The Jacobian matrix evaluated at the second equilibrium point

$E_2 = (-1,-1,1)$ is $J = A|E_2 = (-1,-1,1)$

$$J = \begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.2 & -0.2 & -0.2 \\ -0.01 & -0.01 & -0.1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at equilibrium point E_2 is $|J - mI| = 0$

$$m^3 + 0.7m^2 + 0.058m - 0.0016 = 0 \text{ where } b_1 = 0.7, \quad b_2 = 0.058, \; b_3 = -0.0016$$

Since there is sign changes in the first column of Routh- Hurwitz table, equilibrium point two is unstable.
The Jacobian matrix evaluated at the third equilibrium point $E_3 = (1,1,1)$ is $J = A|E_3 = (1,1,1)$

$$J = \begin{pmatrix} -0.4 & 0.4 & 0 \\ 0.2 & -0.2 & 0.2 \\ 0.01 & 0.01 & -0.1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at equilibrium point E_3 is $|J - mI| = 0$

$$m^3 + 0.7m^2 + 0.058m - 0.0016 = 0$$

where

$$c_1 = 0.7, \quad c_2 = 0.058, \quad c_3 = -0.0016$$

Since there is sign changes in the first column of Routh- Hurwitz table, equilibrium point three is unstable.

Global Stability Analysis of the System

Let $v_1(x,y,z) = \frac{x^2}{0.4} + \frac{y^2}{0.2} + \frac{(z-0.9)^2}{0.1}$ be candidate Lyapunov function at equilibrium point $E_1 = (0,0,0.9)$, then:-

1. $v_1\left(x^*, y^*, z^*\right) = v_1(0,0,0.9) = 0$

2. $v_1(x, y, z) > 0$ for all $(x, y, z) \in D - \left\{\left(x^*, y^*, z^*\right)\right\}$

   Hence, $v_1(x, y, z)$ is positive definite function.

3. $\frac{dv_1}{dt}(x, y, z) = \frac{\partial v_1}{\partial x}(x, y, z)\frac{dx}{dt} + \frac{\partial v_1}{\partial y}(x, y, z)\frac{dy}{dt} + \frac{\partial v_1}{\partial z}(x, y, z)\frac{dz}{dt}$

   $\frac{dv_1}{dt}(x, y, z) = -2g(x, y, z)$

Construct Hessian matrix for $g(x, y, z)$ at the first equilibrium point $E_1 = (0,0,0.9)$ to check
whether Hessian matrix is positive definite.

$$H = \left.\begin{pmatrix} 2 & -1.1z-0.91 & -1.1y \\ -1.1z-0.91 & 2 & -1.1x \\ -1.1y & -1.1x & 2 \end{pmatrix}\right|_{E_1=(0,0,0.9)} = \begin{pmatrix} 2 & -1.9 & 0 \\ -1.9 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

All leading principal minors of Hessian matrix at $E_1 = (0,0,0.9)$ are:-

$$D_1 = 2\ ,\quad D_2 = \begin{vmatrix} 2 & -1.9 \\ -1.9 & 2 \end{vmatrix} = 0.39 \text{ and } D_3 = \begin{vmatrix} 2 & -1.9 & 0 \\ -1.9 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 0.78$$

Since the leading principal minors $D_1 = 2 > 0$, $D_2 = 0.39 > 0$ and $D_3 = 0.78 > 0$, then the Hessian matrix is positive definite. If the Hessian matrix is positive definite, then $\frac{dv_1}{dt}(x, y, z)$ is negative definite function.

4. $\lim\limits_{(x,y,z)\to\infty} v_1(x, y, z) = \lim\limits_{(x,y,z)\to\infty}\left[\frac{x^2}{0.4} + \frac{y^2}{0.2} + \frac{(z-0.9)^2}{0.1}\right] = \infty$

   $v_1(x, y, z)$ is radially unbounded.

As a result, the equilibrium point $E_1 = (0,0,0.9)$ is globally asymptotically stable by Lyapunov stability theorem.

CONCLUSION

In this paper, the Stability and HopfBifurcation analysis of Maxwell-Bloch equations were considered. The system is proved to be dissipative by the aid of divergence test. The result of the study revealed that equilibrium point one is stable and unstable for negative and positive value of pumping energy parameter, respectively. The remaining two equilibrium points are stable and unstable for positive and negative value of pumping energy parameter, respectively. By the aid of Lyapunov theorem, equilibrium point one was proved to be globally asymptotically stable with some specific interval of pumping energy parameter. Furthermore, the result of Hopf bifurcation analysis indicates that the system doesn't undergo Hopf bifurcation at equilibrium point one by any choice of pumping energy parameter. With some specific conditions, the system undergoes Hopf bifurcation about the two remaining equilibrium points for a certain value of pumping energy parameter. Finally, in order to verify the applicability of the result, two numerical examples were solved. MATLAB simulation was also implemented to support the findings of the study.

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