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Fixed Point and Common Fixed Point Results in Dislocated Quasi Metric Spaces

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ABSTRACT

In this paper we establish some new fixed point results for self-mappings satisfying certain contractive conditions in the setting of dislocated quasi-metric spaces. Our established results extend and generalize some existing fixed point results in the literature. We also provide appropriate examples for the usability of the established results.

Keywords: Common fixed point; Complete dq-metric space; Contraction mapping; Fixed point

INTRODUCTION

Fixed point theory is one of the most dynamic research subjects in nonlinear analysis and can be used to many discipline branches such as; control theory, convex optimization, differential equation, integral equation ,economics etc. In this area, the first important and remarkable result was presented by Banach in 1922 for a contraction mapping in a complete metric space. Dass and Gupta (1975) generalized the Banach contraction principle in a metric space for some rational type contractive conditions. Hitzler and Seda (2000) investigated the useful applications of dislocated topology in the context of logic programming semantics. Furthermore, Zeyada *et al*. (2005) generalized the results of Hitzler and Seda (2000) and introduced the concept of complete dislocated quasi metric space.

Aage and Salunke (2008) derived some fixed point theorems in dislocated quasi metric spaces. Similarly, Isufati (2010) proved some fixed point results for continuous contractive condition with rational type expression in the context of a dislocated quasi metric space. Kohli *et al.* (2010) investigated a fixed point theorem which generalized the result of Isufati. In 2012 Zoto (2012) gave some new results in dislocated and dislocated quasi metric spaces. For a continuous self-mapping, a fixed point theorem in dislocated quasi metric spaces was investigated by Shrivastava *et al.* (2012). In 2013, Patel and Patel constructed some new fixed point results in a dislocated quasi metric space. In 2014, Sarwar *et al.* and in 2017 Li *et al.* established fixed point results in dislocated quasi metric spaces.

In the present paper, we establish some fixed point results in the setting of dislocated quasi-metric spaces for single and a pair of continuous self-mappings which extend and generalize the results reported in the cited papers.

MATERIAL AND METHODS

Preliminaries

Definition 2.1. Zeyad *et al*. (2005): Let X be a non-empty set, and let $d : X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

$d_1)$ $d(x,x) = 0;$

$d_2)$ $d(x,y) = d(y,x) = 0,$ implies that $x = y;$

$d_3)$ $d(x,y) = d(y,x)$ for all $x, y \in X;$

$d_4)$ $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X.$

If d satisfies the conditions from d_1 to d_4, then it is called a metric on X, if d satisfies conditions d_2 to d_4, then it is called a dislocated metric (d-metric) on X, and if d satisfies conditions d_2 and d_4, only then it is called a dislocated quasi metric (dq-metric) on X.

Definition 2.2. Zeyad *et al*. (2005): A sequence $\{x_n\}$ in a dq-metric space (X, d) is called Cauchy sequence if for all $\varepsilon > 0$ there exists a positive integer N such that for all $m, n \ge N$, we have $d(x_n, x_m) < \varepsilon$ or $d(x_m, x_n) < \varepsilon.$

Definition 2.3. Zeyad *et al*. (2005): A sequence $\{x_n\}$ is dislocated quasi-converges (dq-converges) to x if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case, x is called a dq-limit of sequence $\{x_n\}$ and we can write

$x_n \to x \ as \ n \to \infty.$

Dentition 2.4 Zeyad *et al*. (2005): A dq-metric space $(X,\ d)$ is called complete if every Cauchy sequence in it is dq-convergent.

Lemma 2.1 Zeyad *et al*. (2005): A dq-limits in a dq-metric space are unique.

Definition 2.5. Let (X, d) be a dq-metric space. A self-map $T : X \to X$ is called contraction if there exists $0 \le k < 1$ such that

$$d(Tx, Ty) \le kd(x, y)$$

for all $x, y \in X$.

Theorem 2.1. Zeyad *et al*. (2005): Let (X, d) be a complete dq-metric space and let $T : X \to X$ be a continuous contraction function. Then T has a unique fixed point in X.

Theorem 2.2. Aage and Salunke (2008): Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying

$$d(Tx, Ty) \le \beta\, [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $0 \le \beta < \frac{1}{2}$. Then T has a unique fixed point.

Theorem 2.3. Aage and Salunke (2008): Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$$

for all $x, y \in X$, where $a_1, a_2, a_3 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + a_3 < 1$. Then T has a unique fixed point.

Theorem 2.4. Shirvastava *et al.* (2012): Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying the following condition:

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 \frac{d(x, Tx) d(y, Ty)}{d(x, y)}$$

for all $x, y \in X$ satisfying $d(x, y) \ne 0$, where $a_1, a_2 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 < 1$. Then T has a unique fixed point.

Theorem 2.5. Shirvastava *et al.* (2012): Let (X, d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping. If

$$d(Tx, Ty) \le a_1 d(x, y) + a_2 \frac{d(x, Tx) d(y, Ty)}{d(x, y)} + a_3[d(x, Tx) + d(y, Ty)] + a_4[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$ satisfying $d(x, y) \ne 0$, where $a_1, a_2, a_3, a_4 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + 2a_3 + 2a_4 < 1$, then T has a unique fixed point.

Theorem 2.6. Zoto *et al* (2012): Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping. If

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_3[d(x,Tx) + d(y,Ty)] + a_4[d(x,Ty) + d(y,Tx)] + a_5[d(x,Tx) + d(x,y)]$$

for all $x, y \in X$ satisfying $d(x,y) \ne 0$,where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + 2a_3 + 2a_4 + 2a_5 < 1,$ then T has a unique fixed point.

Theorem 2.7. Panthi *et al.* (2013): Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping. If

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_3[d(x,Tx) + d(y,Ty)] + a_4[d(x,Ty) + d(y,Tx)] + a_5[d(x,Tx) + d(x,y)] + a_6[d(y,Ty) + d(x,y)]$$

for all $x, y \in X$ satisfying $d(x,y) \ne 0$,where $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 < 1,$ then T has a unique fixed point.

Theorem 2.8. Li *et al.* (2017): Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping. If

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_3[d(x,Tx) + d(y,Ty)] + a_4[d(x,Ty) + d(y,Tx)] + a_5[d(x,Tx) + d(x,y)] + a_6[d(y,Ty) + d(x,y)] + a_7[d(x,Ty) + d(x,y)]$$

for all $x, y \in X$ satisfying $d(x,y) \ne 0,$ where $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1.$ Then T has a unique fixed point.

Theorem 2.9. Patel and Patel (2013): Let (X,d) be a complete dq-metric space and $T : X \to X$ be a continuous self-mapping satisfying

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 d(x,Ty) + a_3 d(y,Tx) + a_4[d(x,Tx) + d(y,Ty)] + a_5[d(x,Ty) + d(y,Tx)]$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$. Then T has a unique fixed point.

Theorem 2.10. Aage and Salunke (2008): Let (X, d) be a complete dq-metric space and $S, T : X \to X$ be two continuous self-mappings satisfying

$$d(Sx, Ty) \le a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty)$$

for all $x, y \in X$, where $a_1, a_2, a_3 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + a_3 < 1$. Then S and T have a unique common fixed point.

Theorem 2.11. Sarwar *et al.* (2014): Let (X, d) be a complete dq-metric space and $S, T : X \to X$ be two continuous self-mappings satisfying

$$d(Sx, Ty) \le a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty) + a_4 [d(x, Sx) + d(y, Ty)] \\ + a_5 [d(x, Ty) + d(y, Sx)]$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}^+$ and $0 \le a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$. Then S and T have a unique common fixed point.

Theorem 2.12. Li *et al.* (2017): Let (X, d) be a complete dq-metric space and $S, T : X \to X$ be two continuous self-mappings satisfying

$$d(Sx, Ty) \le a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty) + a_4 [d(x, Sx) + d(y, Ty)] \\ + a_5 [d(x, Ty) + d(y, Sx)] + a_6 [d(x, Sx) + d(x, y)]$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 and $a_6 \in \mathbb{R}^+$ and

$0 \le a_1 + a_2 + a_3 + 2a_4 + 4a_5 + 2a_6 < 1$.

Then S and T have a unique common fixed point.

RESULTS AND DISCUSSION

Theorem 3.1. Let (X, d) be a complete dislocated quasi metric space and $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ be a continuous self-mapping satisfying the following condition:

$$d(Tx,Ty) \le a_1 d(x,y) + a_2 \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + a_3[d(x,Tx)+d(y,Ty)]$$
$$+a_4[d(x,Ty)+d(y,Tx)] + a_5[d(x,Tx)+d(x,y)] \quad (1)$$
$$+a_6[d(y,Ty)+d(x,y)] + a_7[d(x,Ty)+d(x,y)]$$
$$+a_8[d(y,Tx)+d(x,y)]$$

for all $x, y \in X$ with $d(x,y) \ne 0$, where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \ge 0$ and $0 \le a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 + 3a_8 < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ and define a sequence $\{x_n\}$ in X as follows:

$Tx_n = x_{n+1}$, for $n = 0,1,2,3,\dots$, where $d(x_{n-1}, x_n) \ne 0$. Set $x = x_{n-1}$ and $y = x_n$.

By using (1) , the triangle inequality and $d(x_n, x_n) \le d(x_{n-1}, x_n) + d(x_n, x_{n+1})$

we have

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n) \le a_1 d(x_{n-1},x_n) + a_2 \frac{d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)} + a_3[d(x_{n-1},Tx_{n-1}) + d(x_n,Tx_n)]$$
$$+a_4[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})] + a_5[d(x_{n-1},Tx_{n-1})+d(x_{n-1},x_n)]$$
$$+a_6[d(x_n,Tx_n)+d(x_{n-1},x_n)] + a_7[d(x_{n-1},Tx_n)+d(x_{n-1},x_n)]$$
$$+a_8[d(x_n,Tx_{n-1})+d(x_{n-1},x_n)]$$
$$= a_1 d(x_{n-1},x_n) + a_2 \frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)} + a_3[d(x_{n-1},x_n)+d(x_n,x_{n+1})]$$
$$+a_4[d(x_{n-1},x_{n+1})+d(x_n,x_n)] + a_5[d(x_{n-1},x_n)+d(x_{n-1},x_n)]$$
$$+a_6[d(x_n,x_{n+1})+d(x_{n-1},x_n)] + a_7[d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)]$$
$$+a_8[d(x_n,x_n)+d(x_{n-1},x_n)]$$
$$\le (a_1 + a_3 + 2a_4 + 2a_5 + a_6 + 2a_7 + 2a_8) d(x_{n-1},x_n)$$
$$+(a_2 + a_3 + 2a_4 + a_6 + a_7 + a_8) d(x_n,x_{n+1}).$$

Hence, we have:

$$d\left(x_n,x_{n+1}\right)\le \frac{a_1+a_3+2a_4+2a_5+a_6+2a_7+2a_8}{1-\left(a_2+a_3+2a_4+a_6+a_7+a_8\right)}d\left(x_{n-1},x_n\right).$$

Let

$$\lambda=\frac{a_1+a_3+2a_4+2a_5+a_6+2a_7+2a_8}{1-\left(a_2+a_3+2a_4+a_6+a_7+a_8\right)}.$$

Clearly, $0\le\lambda<1,$ since $0\le a_1+a_2+2a_3+4a_4+2a_5+2a_6+3a_7+3a_8<1.$
So,

$$d\left(x_n,x_{n+1}\right)\le\lambda d\left(x_{n-1},x_n\right).$$

Similarly,

$$d\left(x_{n-1},x_n\right)\le\lambda d\left(x_{n-2},x_{n-1}\right).$$

Thus

$$d\left(x_n,x_{n+1}\right)\le\lambda^2 d\left(x_{n-2},x_{n-1}\right).$$

Continuing the same procedure, we get

$$d\left(x_n,x_{n+1}\right)\le\lambda^n d\left(x_0,x_1\right).$$

Now, for any $m,n\in\mathbb{N}$ with $m>n,$ using the triangle inequality, we get

$$\begin{aligned}d\left(x_n,x_m\right)&\le d(x_n,x_{n+1})+d\left(x_{n+1},x_{n+2}\right)+\ldots+d\left(x_{m-1},x_m\right)\\&\le\lambda^n d\left(x_0,x_1\right)+\lambda^{n+1}d\left(x_0,x_1\right)+\ldots+\lambda^{m-1}d\left(x_0,x_1\right)\\&=(\lambda^n+\lambda^{n+1}+\lambda^{n+2}+\ldots+\lambda^{m-1})d\left(x_0,x_1\right)\\&\le\frac{\lambda^n}{1-\lambda}d\left(x_0,x_1\right).\end{aligned}$$

For any $\varepsilon>0,$ we can choose a positive integer N such that, $\frac{\lambda^N}{1-\lambda}d\left(x_0,x_1\right)<\varepsilon.$

It follows that for any $m,n\ge N,$ we have

$$d\left(x_n,x_m\right)\le\frac{\lambda^n}{1-\lambda}d\left(x_0,x_1\right)\le\frac{\lambda^N}{1-\lambda}d\left(x_0,x_1\right)<\varepsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi metric space $\left(X,d\right).$ So, there exists $u\in X$ such that $\lim\limits_{n\to\infty}x_n=u.$ Since T is continuous, so we have

$$Tu=T(\lim_{n\to\infty}x_n)=\lim_{n\to\infty}\mathrm{T}(x_n)=\lim_{n\to\infty}x_{n+1}=u.$$

Hence, u is a fixed point of $T.$

Uniqueness: Suppose that T has two distinct fixed points u and v. Condition (1) implies that $d(u,u)=0$ and $d(v,v)=0$.

If $d(u,u)>0$ and $d(v,v)>0$, then by condition (1) we have

$$d(u,u)=d(Tu,Tu)\le a_1 d(u,u)+a_2\frac{d(u,Tu)d(u,Tu)}{d(u,u)}+a_3[d(u,Tu)+d(u,Tu)]$$
$$+a_4[d(u,Tu)+d(u,Tu)]+a_5[d(u,Tu)+d(u,u)]$$
$$+a_6[d(u,Tu)+d(u,u)]+a_7[d(u,Tu)+d(u,u)]$$
$$+a_8[d(u,Tu)+d(u,u)]$$
$$=(a_1+a_2+2a_3+2a_4+2a_5+2a_6+2a_7+2a_8)d(u,u)<d(u,u),$$

Since, $a_1+a_2+2a_3+2a_4+2a_5+2a_6+2a_7+2a_8<1$, which is a contradiction.
Hence, $d(u,u)=0$.
Similarly, $d(v,v)=0$.
To show $d(u,v)=d(v,u)=0$ using condition (1) we have

$$d(Tu,Tv)\le a_1 d(u,v)+a_2\frac{d(u,Tu)d(v,Tv)}{d(u,v)}+a_3[d(u,Tu)+d(v,Tv)]$$
$$+a_4[d(u,Tv)+d(v,Tu)]+a_5[d(u,Tu)+d(u,v)] \qquad (2)$$
$$+a_6[d(v,Tv)+d(u,v)]+a_7[d(u,Tv)+d(u,v)]$$
$$+a_8[d(v,Tu)+d(u,v)].$$

Finally, from (2) we get:
$$d(u,v)\le(a_1+a_4+a_5+a_6+2a_7+a_8)d(u,v)+(a_4+a_8)d(v,u). \qquad (3)$$

Similarly, we have:
$$d(v,u)\le(a_1+a_4+a_5+a_6+2a_7+a_8)d(v,u)\ +(a_4+a_8)d(u,v). \qquad (4)$$

Subtracting (4) from (3) we have:
$$|d(u,v)-d(v,u)|\ \le\ (a_1+a_5+a_6+2a_7)|d(u,v)-d(v,u)|\ . \qquad (5)$$

Since $0\le\ (a_1+a_5+a_6+2a_7)<1$, so the inequality (5) is possible if

$$d(u,v)-d(v,u)=0. \qquad (6)$$

Taking equations (3), (4) and (6) into account, we have $d(u,v)=0$ and $d(v,u)=0.$
Thus by condition (d_2), $u=v.$ Hence T has a unique fixed point in X.

Example 3.1. Let $X=[0,1]$ with a complete dq-metric defined by
$d(x,y)=\ |x|$, for all $x,y\in X$ and define a continuous self-mapping T by $Tx=\dfrac{x}{2}$.
With
$a_1=\frac{1}{8}, a_2=\frac{1}{12}, a_3=\frac{1}{14}, a_4=\frac{1}{32}, a_5=\frac{1}{20}, a_6=\frac{1}{16}, a_7=\frac{1}{18}$ *and* $a_8=\frac{1}{24}$,
T satisfies all the conditions of Theorem 3.1 and $x=0$ is the unique fixed point of T in X.

Remarks: In the Theorem 3.1:

If $a_8=0$, then we get Theorem 2.8.
If $a_7=a_8=0$, then we get Theorem 2.7.

If $a_6=a_7=a_8=0$, then we get Theorem 2.6.
If $a_5=a_6=a_7=a_8=0$, then we get Theorem 2.5.
If $a_3=a_4=a_5=a_6=a_7=a_8=0$, then we get Theorem 2.4.

In the following we prove a common fixed point result.

Theorem 3.2. Let (X,d) be a complete dq-metric space and $S,T:X\to X$ be two continuous self-mappings satisfying

$$d(Sx,Ty)\le a_1d(x,y)+a_2d(x,Sx)+a_3d(y,Ty)+a_4[d(x,Sx)+d(y,Ty)]$$
$$+a_5[d(x,Ty)+d(y,Sx)]+a_6[d(x,Sx)+d(x,y)]+a_7[d(y,Ty)+d(x,y)] \qquad (2)$$

for all $x,y\in X$, where $a_1,a_2,a_3,a_4,a_5,a_6,a_7\in\mathbb{R}^+$ and
$0\le a_1+a_2+a_3+2a_4+4a_5+2a_6+2a_7<1.$
Then S and T have a unique common fixed point.

Proof: Let $x_0\in X$, we define a sequence $\{x_n\}$ in X as follows:

$$Sx_0 = x_1,\ Sx_1 = x_2,\ \ldots,\ Sx_{2n} = x_{2n+1}$$

and

$$Tx_1 = x_2,\ Tx_2 = x_3,\ \ldots,\ Tx_{2n-1} = x_{2n}$$

For all $n \in \mathbb{N}$ we have

$$d\left(x_{2n+1}, x_{2n+2}\right) = d\left(Sx_{2n}, Tx_{2n+1}\right).$$

By using (2), the triangle inequality and $d\left(x_n, x_n\right) \le d\left(x_{n-1}, x_n\right) + d\left(x_n, x_{n+1}\right)$

We have,

$$\begin{aligned}
d\left(x_{2n+1}, x_{2n+2}\right) &= d\left(Sx_{2n}, Tx_{2n+1}\right) \\
&\le a_1 d(x_{2n}, x_{2n+1}) + a_2 d\left(x_{2n}, Sx_{2n}\right) + a_3 d\left(x_{2n+1}, Tx_{2n+1}\right) \\
&\quad + a_4[d\left(x_{2n}, Sx_{2n}\right) + d\left(x_{2n+1}, Tx_{2n+1}\right)] + a_5[d\left(x_{2n}, Tx_{2n+1}\right) + d\left(x_{2n+1}, Sx_{2n}\right)] \\
&\quad + a_6[d\left(x_{2n}, Sx_{2n}\right) + d\left(x_{2n}, x_{2n+1}\right)] + a_7[d\left(x_{2n+1}, Tx_{2n+1}\right) + d\left(x_{2n}, x_{2n+1}\right)] \\
&= a_1 d(x_{2n}, x_{2n+1}) + a_2 d\left(x_{2n}, x_{2n+1}\right) + a_3 d\left(x_{2n+1}, x_{2n+2}\right) \\
&\quad + a_4[d\left(x_{2n}, x_{2n+1}\right) + d\left(x_{2n+1}, x_{2n+2}\right)] + a_5[d\left(x_{2n}, x_{2n+2}\right) + d\left(x_{2n+1}, x_{2n+1}\right)] \\
&\quad + a_6[d\left(x_{2n}, x_{2n+1}\right) + d\left(x_{2n}, x_{2n+1}\right)] + a_7[d\left(x_{2n+1}, x_{2n+2}\right) + d\left(x_{2n}, x_{2n+1}\right)] \\
&\le (a_1 + a_2 + a_4 + 2a_5 + 2a_6 + a_7) d\left(x_{2n}, x_{2n+1}\right) + (a_3 + a_4 + 2a_5 + a_7) d\left(x_{2n+1}, x_{2n+2}\right),
\end{aligned}$$

which implies that:

$$d\left(x_{2n+1}, x_{2n+2}\right) \le \frac{(a_1 + a_2 + a_4 + 2a_5 + 2a_6 + a_7)}{(1 - a_3 + a_4 + 2a_5 + a_7)} d\left(x_{2n}, x_{2n+1}\right).$$

Let

$$\lambda = \frac{a_1 + a_2 + a_4 + 2a_5 + 2a_6 + a_7}{1 - (a_3 + a_4 + 2a_5 + a_7)}.$$

Clearly, $0 \le \lambda < 1$, since $0 \le a_1 + a_2 + a_3 + 2a_4 + 4a_5 + 2a_6 + 2a_7 < 1.$

So,

$$d\left(x_{2n+1}, x_{2n+2}\right) \le \lambda d\left(x_{2n}, x_{2n+1}\right).$$

Similarly,

$$d\left(x_{2n}, x_{2n+1}\right) \le \lambda d\left(x_{2n-1}, x_{2n}\right).$$

Thus

$$d\left(x_{2n+1}, x_{2n+2}\right) \le \lambda^2 d\left(x_{2n-1}, x_{2n}\right).$$

Continuing the same procedure, we get

$$d\left(x_n, x_{n+1}\right) \le \lambda^n d\left(x_0, x_1\right).$$

Now, for any $m, n \in \mathbb{N}$ with $m > n$, using the triangle inequality, we get

$$\begin{aligned} d\left(x_n, x_m\right) &\le d(x_n, x_{n+1}) + d\left(x_{n+1}, x_{n+2}\right) + \ldots + d\left(x_{m-1}, x_m\right) \\ &\le \lambda^n d\left(x_0, x_1\right) + \lambda^{n+1} d\left(x_0, x_1\right) + \ldots + \lambda^{m-1} d\left(x_0, x_1\right) \\ &= (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \ldots + \lambda^{m-1}) d\left(x_0, x_1\right) \\ &\le \frac{\lambda^n}{1-\lambda} d\left(x_0, x_1\right). \end{aligned}$$

For any $\varepsilon > 0$, we can choose a positive integer N such that, $\frac{\lambda^N}{1-\lambda} d\left(x_0, x_1\right) < \varepsilon.$

For any $m, n \ge N$, we have

$$d\left(x_n, x_m\right) \le \frac{\lambda^n}{1-\lambda} d\left(x_0, x_1\right) \le \frac{\lambda^N}{1-\lambda} d\left(x_0, x_1\right) < \varepsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence in a complete dislocated quasi metric space $\left(X, d\right)$. So, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u.$ Also the sub-sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to $u.$ Since T is a continuous mapping, therefore

$$\lim_{n\to\infty} x_{2n+1} = u \Rightarrow \lim_{n\to\infty} Tx_{2n+1} = Tu \Rightarrow \lim_{n\to\infty} x_{2n+2} = Tu$$

Hence,

$$Tu = u.$$

Similarly, taking the continuity of S, we can show that $Su = u.$ Hence u is the common fixed point of S and T.

Uniqueness: Suppose that S and T have two distinct common fixed points u and v.

Consider

$$\begin{aligned} d\left(Su, Tv\right) \le\ & a_1 d\left(u, v\right) + a_2 d\left(u, Su\right) + a_3 d\left(v, Tv\right) + a_4 [d\left(u, Su\right) + d\left(v, Tv\right)] \\ & + a_5 [d\left(u, Tv\right) + d\left(v, Su\right)] + a_6 [d\left(u, Su\right) + d\left(u, v\right)] + a_7 [d\left(v, Tv\right) + d\left(u, v\right)] \end{aligned} \quad (3)$$

Since u and v are common fixed points of S and T, condition (2) implies that $d\left(u, u\right) = 0$ and $d\left(v, v\right) = 0.$

Finally, from (3) we get

$$d(u,v) \le (a_1 + a_5 + a_6 + a_7)d(u,v) + (a_5)d(v,u). \tag{4}$$

Similarly, we have:

$$d(v,u) \le (a_1 + a_5 + a_6 + a_7)d(u,v) + (a_5)d(u,v). \tag{5}$$

Subtracting (5) from (4) we have

$$|d(u,v) - d(v,u)| \le (a_1 + a_6 + a_7)|d(u,v) - d(v,u)|. \tag{6}$$

Since $0 \le a_1 + a_6 + a_7 < 1$, so the above inequality (6) is possible if

$$d(u,v) - d(v,u) = 0. \tag{7}$$

Taking equations (4), (5) and (7) into account, we have $d(u,v) = 0$ and $d(v,u) = 0$. Thus by condition (d_2) we have $u = v$. Hence S and T have a unique common fixed point in X.

Example 3.2. Let $X = [0,1]$ and a complete dq-metric defined by $d(x,y) = |x|$, for all $x, y \in X$ and define a continuous self-mappings S and T by

$Sx = 0.\ Tx = \frac{x}{8}.$

With $a_1 = \frac{1}{8}, a_2 = \frac{1}{12}, a_3 = \frac{1}{16}, a_4 = \frac{1}{18}, a_5 = \frac{1}{32}, a_6 = \frac{1}{20}$ and $a_7 = \frac{1}{24}$, S and T satisfy all the conditions of Theorem 3.2 and $x = 0$ is the unique common fixed point of S and T in X.

Remarks: In the Theorem 3.2:

If $a_7 = 0$, then we get Theorem 2.12.

If $a_6 = a_7 = 0$, then we get Theorem 2.11.

If $S = T$ and $a_6 = a_7 = 0$, then we get Theorem 2.9.

If $a_4 = a_5 = a_6 = a_7 = 0$, then we get Theorem 2.10.

If $S = T$ and $a_4 = a_5 = a_6 = a_7 = 0$, then we get Theorem 2.3.

If $S = T$ and $a_1 = a_2 = a_3 = a_5 = a_6 = a_7 = 0$, then we get Theorem 2.2.

CONCLUSION

The derived results extend and generalize theorems from Theorem 2.2 to Theorem 2.12 in the setting of dislocated quasi metric spaces.

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