ORIGINAL ARTICLE

Investigation of Nonholonomic Mechanics, Vakonomic Mechanics and Chetaev Method in Modeling Constrained Dynamic Systems

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Abstract:
In this article, methods of modeling dynamic systems namely, Nonholonomic mechanics, Vakonomic mechanics and Chetaev methods for constrained dynamic system are investigated. The fact that Vakonomic mechanics gives a different motion equation to the other methods is verified using a particular example. It is shown that the three methods give the same motion equation for holonomic system. For nonholonomic system, the Vakonomic dynamics gives a different motion equation to the others. Moreover, Chetaev equation is proved without using Chetaev condition. A particular example is provided in verifying that Chetaev condition is not always valid. Finally, the reason why the Vakonomic mechanics gives a different motion equation in the case of nonholonomic system is scrutinized based on the definition of Vakonomic mechanics whose motion equation is obtained through a purely variational principle. An example is given to strengthen the arguments.

*Key Terms***:** *Nonholonomic mechanics, Vakonomic mechanics, Chetaev formula, holonomic constraint, semi-holonomic constraints, nonholonomic constraints, Variational principles, Infinitesimal variations*

Introduction:
The term, Borisov A.V. et al(2002), Manuel de León.(2012) "nonholonomic system" was coined by Hertz (1894).

Nonholonomic systems are, mechanical systems with constraints on their velocities that are not derivable from position constraints.

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Nonholonomic systems arise in mechanical systems that have rolling contact (rolling of wheels without slipping) or certain kinds of sliding contact such as the sliding of skates. One of the first, Borisov A.V. et al(2002), discoveries by Hertz was that, the usual integral variational principles such as the principle of least action or Hamilton's principle do not hold for nonholonomic systems. Moreover, Hertz has classified Lagrangian system with linear constraints into holonomic and nonholonomic according to whether the imposed constraints are holonomic or not.

The motion equations resulting from classical nonholonomic mechanics using the Lagrange−D'Alembert principle is not given in the form of a variational problem. That is, the motion of the nonholonomic system is not a critical point of any functional in the sense of the Calculus of Variations Kozlov (1983).

In an attempt to circumvent this difficulty, Kozlov (1983), Arnold V.I.(1988), gives a variational formulation of constrained motion, calling the resulting equations of "variational axiomatic kind," leading to the name "vakonomic."

One of the more interesting historical events Borisov A.V. et al(2002), Manuel de León.(2012), Абрамов Н.В. et al(2013), Bloch A.M.(2003) of nonholonomic systems is related to the paper by Korteweg (1899) .Up to that point there was some confusion in the literature between nonholonomic mechanical systems and variational nonholonomic systems. One of the purposes of Korteweg's paper was to uncurl this misunderstanding. Accordingly the difference between nonholonomic and vakonomic dynamics relies in the different principle applied in both cases:

- Nonholonomic dynamics is derived using the d' Alembert principle.
- Vakonomic dynamics is obtained using a variational principle looking for external curves among those satisfying the constraints.

The motion equations resulting from vakonomic and classical nonholonomic mechanics are genuinely different from each other as is shown in Lewis et.al(1995). Moreover, Kozlov himself made it clear that the equations obtained through vakonomic dynamics and classical nonholonomic mechanics are the same in the case of holonomic constraints and for nonintegrable constraints, vakonomic mechanics gives a different motion equation Kozlov V.V.(1982a) to classical nonholonomic mechanics.

In several articles Arnold V.I et al (1988), Lewis A. et.al (1995), Kozlov V.V.(1982a) the differences of motion equations resulting from vakonomic mechanics and classical nonholonomic mechanics are shown using experimentation of different particular examples. This paper gives a theoretical background showing that:

1. The two mechanics give the same motion equations for holonomic systems.
2. The two mechanics give different motion equations for first order linear nonholonomic systems**.**

In addition, in several literatures, the reason why the two mechanics give different motion equations is not addressed. This paper gives an answer to:

3. The question "*why do these methods give different motion equations*?" is scrutinized.
4. Moreover, in this paper, Chetaev equation is proved without using Chetaev condition.

The organization of the paper is, in section 2, motion equations resulting from nonholonomic mechanics, Chetaev method and vakonomic mechanics are revisited. A particular example showing that the vakonomic mechanics gives a different motion equation from that of nonholonomic mechanics and Chetaev method is given. In section 3, an investigation of vakonomic mechanics for the cases of holonomic constraints is made.

In section 4, Chetaev equation is proved without using Chetaev condition. An example where Chetaev condition doesn't hold is provided. Moreover, the reason why vakonomic mechanics gives different equation of motions to nonholonomic mechanics is scrutinized.

Nonholonomic and Vakonomic Dynamics Revisited
Variational Principles for Constrained Systems

Let Q be a configuration manifold Абрамов Н.В. et al(2013), Bloch A.M (2003), with dimension n and TQ its tangent bundle. Denote by $q^A, A = 1, 2, \ldots n.$ the coordinates on Q and by $(q^A, \dot{q}^A)$ the induced coordinates on TQ. Define the regular mechanical Lagrangian as:

$L: TQ \to \mathbb{R}: L = T - V$, where, $T: TQ \to \mathbb{R}$ is the kinetic energy of the system, $V: Q \to \mathbb{R}$ is the potential energy of the system. $\mathbb{R}$ – the set of real numbers.

A set of twice differentiable curves connecting two given points q_1 and q_2, denoted by $C^2(q_1, q_2, \text{[a,b]})$, in Q is define on an interval $[a, b]$ as:

$C^2(q_1, q_2, \text{[a,b]}) = \{c: [a, b] \to Q \mid c$ is

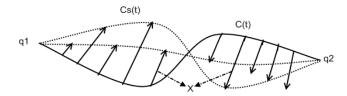a C^2 curve, $c(a) = q_1$ and $c(b) = q_2\}$ and is called the path space from q_1 to q_2.(See Fig.1).

Fig.1. Variations Cs and infinitesimal varitions X of a curve C with end points q1 and q2

This set is a differentiable infinite-dimensional manifold Bloch A.M (2003), Мухарлямов Р.Г. (2013).The tangent space to $C^2(q_1, q_2, \text{[a, b]})$ at curve $c \in C^2(q_1, q_2, \text{[a, b]})$ is given as :

$T_c C^2(q_1, q_2, \text{[a,b]}) = \{X : [a, b] \to TQ \mid X$ is a C^2, map[1] $\tau_Q \circ X = c$ and

$X(a) = X(b)\}$

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[1] $\tau_Q: TQ \to Q$ is a map that takes a tangent vector X to the curve $c \in Q$ at which the vector X is attached (that is, $X \in T_c M$).The inverse image $\tau_M^{-1}(c)$ of a curve $c \in Q$

Since X is a tangent vector to the manifold $C^2(q_1, q_2, [a, b]$ we may write it as the tangent vector at $S = 0$ of a curve in $C^2(q_1, q_2, [a, b], s \in (-\mathcal{E}, \mathcal{E}) \subset \mathbb{R} \longrightarrow C_s$ which passes through C at $S = 0$, i.e. $c_0 = c$, as:

$$X = \frac{dc_s}{ds}|_{s=0} \in T_{c(t)}Q . \qquad (2.1)$$

Given the Lagrangian function, two fixed points $q_0, q_1 \in Q$ and a fixed time interval $[a, b]$, the associated *action integral* is the Мухарлямов Р.Г. (2013), Cortés J. (2002) real valued map given by:

$\mathcal{J} = C^2(q_1, q_2, [\text{a, b}]) \longrightarrow \mathbb{R}$ defined by:

$$\mathcal{J}(c) = \int_a^b L(c^{(1)}(t))dt = \int_a^b L(c(t), c'(t))dt . \qquad (2.2)$$

Theorem1.Hamilton's variation principles, Мухарлямов Р.Г. (2013).

A curve $c \in C^2(q_1, q_2)$ is a motion of the Lagrangian system defined by L if and only if C is a critical point of the action integral $\mathcal{J}$, (i.e., $\frac{d}{dt}\mathcal{J}(c) = 0$).

2.2 Modeling Mechanical Systems

Definition 1: A curve $c \in C^2(q_1, q_2, [\text{a, b}])$, is called, Мухарлямов Р.Г. (2013),a *critical point* to the action integral $\mathcal{J}$ if and only if $d\mathcal{J}(c) \cdot X = 0$ for every $X \in T_c C^2(q_1, q_2, [\text{a,b}])$.

It is convenient to write $d\mathcal{J}(c) \cdot X$ as:

$$d\mathcal{J}(c) \cdot X = \frac{d}{ds}|_{s=0}\mathcal{J}(c_s) = \int_a^b \frac{d}{ds}|_{s=0}L(c_s, \dot{c}_s)dt. \qquad (2.3)$$

Equation (2.3), using the end point conditions $\delta c_s|_{t=a} = \delta c_s|_{t=b} = 0$, the commutation $\delta\dot{c}_s = \frac{d}{dt}\delta c_s$ and method of integration by parts leads to:

$$d\mathcal{J}(c).X = \int_a^b \left(\frac{\partial L}{\partial q^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A}\right)X^A dt = 0 \quad ,A = 1 \ 2 \ ...n, \qquad (2.4)$$

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under the natural projection τ_Q is the tangent space $T_{c(t)}Q$. One refers to X as an *infinitesimal variation* (the set of all virtual variations) of the curve c subject to fixed endpoints. Classically, the notation $X = \delta c$ is used.

$\forall X \in T_c C^2(q_1, q_2, \text{[a, b]})$ and $\forall c \in C^2(q_1, q_2, [a, b])$.

Note also that as can be seen from (2.4), the nonholonomic constrained variational problem does not immediately give the required equations of motion. This task is taken up in the following cases for nonholonomic and vakonomic mechanics in that order.

Modeling Nonholonomic Mechanics
In this section, we derive the equations of motion for nonholonomic systems subject to affine constraints.
A nonholonomic Lagrangian, Lewis et al(1995), Мухарлямов Р.Г. (2013) system on a manifold Q consists of a pair (L, B). L is the Lagrangian of the system and B is a submanifold of TQ.

The allowed velocities for the nonholonomic Lagrangian system are those belonging to B. We assume B is an affine sub bundle modeled on a vector bundle D. Being an affine, there exists a vectorfield η on Q such that $v_q \in B_q$ if and only if $v_q - \eta(q) \in D_q$ The fulfilling, Мухарлямов Р.Г. (2013), of the constraints requires the introduction of some unknown reaction forces. In connection with the problem of eliminating this unknown character, it is customary to introduce the concept of virtual displacement. Let us consider a first order nonintegrable linear nonholonomic velocity constraint. In this case the constraints can be expressed in the form:

$$\phi^a(q, \dot{q}, t) = \mu_{aA}(q, t)\dot{q}^A + \mu_a(q, t) = 0, \qquad a = 1, \dots, k. \qquad (2.5)$$

In contrast to holonomic constraints, the nonintegrable constraint (2.5) directly restricts the kinematically possible velocities and therefore cannot be directly embedded in $L(q, \dot{q}, t)$ in order to reduce the number n of generalized coordinates to $n - k$ independent coordinates. But since virtual displacements δq coincide with possible displacements dq in the limit of frozen constraints ($\delta q = 0$), they satisfies the linear set of conditions,

$$\mu_{aA}(q, t)\delta q^A = 0. \qquad (2.6)$$

This can be adjoined to expression (2.4) and results in:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} - \frac{\partial L}{\partial q^A} = \lambda^a \mu_{aA}. \qquad (2.7)$$

Where, $\lambda^a, a = 1, ..., k,$are Lagrange multipliers, $A = 1, ... n$ and

$\mu_{aA}(q, t) = \dfrac{\partial \phi^a}{\partial \dot{q}^A}$ is Jacobian matrix of the constraints. The equation of motion

including external forces F_A and the constraints is given by:

$$\begin{cases} \dfrac{\partial L}{\partial q^A} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^A} = \lambda^\alpha \mu_{\alpha A} + F_A, \quad \alpha = 1,2,...,k \\ \mu_{\alpha A}(q)\dot{q}^A + \mu_{\alpha 0}(q) = 0 \end{cases} \qquad (2.8)$$

If in instead of constraints defined by (2.5), the constraints are more generically defined by the vanishing of the set of k independent maps $\phi^\alpha : TQ \longrightarrow \mathbb{R}$ and, $\phi^\alpha(q^A, \dot{q}^A) = 0$, $1 \le \alpha \le k$, determine locally the nonlinear sub-manifold B,then the Chetaev rule implies that the equations of motion for a constrained Lagrangian system, instead of (2.7), becomes:

$$\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^A} - \dfrac{\partial L}{\partial q^A} = \lambda^\alpha \dfrac{\partial \phi^\alpha}{\partial \dot{q}^A} . \qquad (2.9)$$

Equation (2.9) together with the constraint equations (2.5) is called **Chetaev equations.**

Note that, in Chetaev equation the virtual displacement equation is given by the Chetaev condition: $\dfrac{\partial \phi^\alpha}{\partial \dot{q}^A} . \delta q^A = 0$ in steady of (2.6).

Modeling Vakonomic Mechanics

The Lagrange-D'Alembert principle is not given in the form of a variational problem. That is, the motion of the nonholonomic system resulting from equation (2.8) is not a critical point of any functional in the sense of the Calculus of Variations Kozlov V.V.(1982a).

In an attempt to circumvent this Arnold V.I. (1985), difficulty, Kozlov V.V.(1983) gives a variational formulation of constrained motion, calling the resulting equations of "variational axiomatic kind," leading to the name "vakonomic".

Definition2: The vakonomic problem consists of extermizing the functional J defined by equation (2.2) among the curves satisfying, Lewis et al (1995), the constraints. Hence a curve $c \in C^2(q_1, q_2, \ [a,b], B)$, is a solution of the vakonomic problem if C is a critical point of $J|_{C^2(q_1, q_2, [a,b], B)}$

We may use the Hamilton's principle, where by the motions of the system are extremal of the variational problem of Lagrange. The equation of motion then can be obtained as the Euler-Lagrange equations for an extended singular Lagrangian**:**

$$\mathcal{L} = TQ \times \mathbb{R}^k \longrightarrow \mathbb{R}, \ \mathcal{L} = L - \lambda^\alpha \phi^\alpha$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}^A}\right) - \dfrac{\partial \mathcal{L}}{\partial q^A} = 0, & A = 1.2,..,n \\[3mm] \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\lambda}^\alpha}\right) - \dfrac{\partial \mathcal{L}}{\partial \lambda^\alpha} = \phi^\alpha, & \alpha = 1,2,...,k \end{cases} \qquad (2.10)$$

Equation (2.10) can equivalently, Lewis et al(1995),be expressed in the form:

$$\begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^A} - \dfrac{\partial L}{\partial q^A} = \dot{\lambda}^\alpha \dfrac{\partial \phi^\alpha}{\partial \dot{q}^A} + \lambda^\alpha \left[\dfrac{d}{dt}\left(\dfrac{\partial \phi^\alpha}{\partial \dot{q}^A}\right) - \left(\dfrac{\partial \phi^\alpha}{\partial q^A}\right)\right], 1 \le A \le n \\[3mm] \phi^\alpha(q^A, \dot{q}^A) = 0, 1 \le \alpha \le k \end{cases} \qquad (2.11)$$

Where, λ^α are the Lagrange multipliers.

The Rolling Sphere.

In this section an example showing that the vakonomic mechanics gives a different motion equation to that of nonholonomic mechanics and Chetaev method is given.

Example1.

Consider a homogeneous sphere rolling on a plane. Let the plane rotate with constant angular velocity Ω about the Z-axis and $\omega = (\omega_x, \omega_y, \omega_z)$ be the angular velocity vector of the sphere measured with respect to the inertial frame.

Let m be the mass of the sphere, mk^2 — its inertia about any axis, and let a be its radius. The configuration space is $Q = R^2 \times SO(3)$, $(x, y, a) -$ denotes the position of the center of the sphere and $(\varphi, \theta, \psi) -$ denote the Eulerian angles.

The contact condition in terms of the coordinate $W \triangleq (x, y, a)$ of the center of the sphere, the angular velocity Ω of the rotating table (pointing upward) and that of the ball ω may be written as:

$$\phi = (\phi^1, \phi^2) = \vec{V} + \omega \times (-a\vec{k}) = \Omega\vec{k} \times W,$$

and is equivalent to:

$$\begin{cases} \phi^1 = \dot{x} - a\omega_y + y\Omega = 0 \\ \phi^2 = \dot{y} + a\omega_x - x\Omega = 0 \end{cases} \qquad (2.12)$$

Let us consider the dynamic equation of the system using the different mechanics.

I.   Nonholonomic mechanics equation of motion of the rolling Sphere

The Lagrangian of the system is given by:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mk^2(\omega_x{}^2 + \omega_y{}^2 + \omega_z{}^2).$$

Based on equation (2.7) we have:

$$\lambda^{\alpha}\mu_{\alpha A} = \begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & -a & 0 \\ 0 & 1 & a & 0 & 0 \end{pmatrix}. \tag{2.13}$$

We use quasi-velocities $\dot{\varphi} = (\dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3)$ such that

$$(\dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3) \triangleq \omega = (\omega_x, \omega_y, \omega_z).$$

Let us define momenta:

$$\begin{cases} p_x = \dfrac{\partial L}{\partial \dot{x}} = m\dot{x} \\[2mm] p_y = \dfrac{\partial L}{\partial \dot{y}} = m\dot{y} \\[2mm] p_{\varphi_1} = \dfrac{\partial L}{\partial \dot{\varphi}_1} = mk^2 \omega_x \\[2mm] p_{\varphi_2} = \dfrac{\partial L}{\partial \dot{\varphi}_2} = mk^2 \omega_y \end{cases} \tag{2.14}$$

Then from (2.13) and (2.14) we obtain:

$$\dot{p}_x = \lambda^1, \dot{p}_y = \lambda^2, \dot{p}_{\varphi_1} = a\lambda^2, \dot{p}_{\varphi_2} = -a\lambda^1.$$

Setting an initial condition of

$$p_x(0) = 0, p_y(0) = 0, p_{\varphi_1}(0) = 0, p_{\varphi_2}(0) = 0,$$ we obtain:

$$\begin{cases} p_x = \lambda^1 t \\ p_y = \lambda^2 t \\ p_{\varphi_1} = a\lambda^2 t \\ p_{\varphi_2} = -a\lambda^1 t \end{cases} \tag{2.15}$$

From (2.14) and (2.15) we obtain:

$$\begin{cases} m\dot{x} = \lambda^1 t \\ m\dot{y} = \lambda^2 t \\ mk^2 \omega_x = a\lambda^2 t \\ mk^2 \omega_y = -a\lambda^1 t \end{cases} \tag{2.16}$$

From equation (2.16), it follows that: $\dfrac{mk^2 \omega_x}{at} = \lambda^2$ and $\dfrac{-mk^2 \omega_y}{at} = \lambda^1$.

Using these values of λ^1, λ^2 , the constraint equation (2.12) and (2.16) we have the dynamic equation of the system given by:

$$\begin{cases} \dot{x} = \mu y \\ \dot{y} = -\mu x, \end{cases} \tag{2.17}$$

where $\mu \triangleq \dfrac{k^2 \Omega}{a^2 + k^2}$

II. Vakonomic mechanics equation of motion of the rolling Sphere

The extended Lagrangian is given by:

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mk^2\left(\omega_x{}^2 + \omega_y{}^2 + \omega_z{}^2\right) - \lambda^1\left(\dot{x} - a\omega_y + y\Omega\right)$$
$$- \lambda^2\left(\dot{y} + a\omega_x - x\Omega\right)$$

Let us define momenta:

$$
\begin{cases}
p_x = \dfrac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} - \lambda^1 \\[2mm]
p_y = \dfrac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} - \lambda^2 \\[2mm]
p_{\varphi_1} = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = mk^2\omega_x - a\lambda^2 \\[2mm]
p_{\varphi_2} = \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = mk^2\omega_y + a\lambda^1
\end{cases}
\qquad (2.18)
$$

From (2.10) and (2.18) we have:

$$\dot{p}_x = -\lambda^2\Omega, \quad \dot{p}_y = -\lambda^1\Omega, \quad \dot{p}_{\varphi_1} = 0, \quad \dot{p}_{\varphi_2} = 0.$$

Setting an initial condition of

$$p_x(0) = 0, p_y(0) = 0, p_{\varphi_1}(0) = 0, p_{\varphi_2}(0) = 0,\text{ we obtain:}$$

$$
\begin{cases}
p_x = -\lambda^2\Omega t \\[1mm]
p_y = -\lambda^1\Omega t \\[1mm]
mk^2\omega_x = a\lambda^2 \\[1mm]
mk^2\omega_y = -a\lambda^1
\end{cases}
\qquad (2.19)
$$

From equations (2.18) and (2.19) it follows that:

$$\begin{cases} m\dot{x} - \lambda^1 = -\lambda^2 \Omega t \\ m\dot{y} - \lambda^2 = -\lambda^1 \Omega t \\ \quad mk^2 \omega_x = a\lambda^2 \\ \quad mk^2 \omega_y = -a\lambda^1 \end{cases} \qquad (2.20)$$

From (2.20) we obtain $\dfrac{mk^2\omega_x}{a} = \lambda^2$ and $\dfrac{-mk^2\omega_y}{a} = \lambda^1$. Using these values of

λ^1, λ^2, the values $\dfrac{\dot{x}+y\Omega}{a} = \omega_y$, $\dfrac{x\Omega-\dot{y}}{a} = \omega_x$ and the constraint equations (2.12),

we have the dynamic equation given by:

$$\begin{cases} \dot{x} = \dfrac{(\mu^2 - \mu\Omega)tx + ((\mu t)^2\Omega - \mu)y}{1 - (\mu t)^2} \\ \dot{y} = \dfrac{(\mu - (\mu t)^2\Omega)x - (\mu^2 - \mu\Omega)ty}{1 - (\mu t)^2} \end{cases}, \qquad (2.21)$$

where $\mu = \dfrac{k^2\Omega}{a^2 + k^2}$

One can observe that, motion equations (2.17) and (2.21) are in general not equal to each other. Moreover using Chetaev equation (2.9) one can easily show that, motion equation of the rolling sphere is identically equal to the motion equation given by (2.21) obtained from nonholonomic mechanics, equation (2.17).

Vakonomic Mechanics in the Case of Holonomic Constraints
In the vakonomic mechanics, Kozlov V.V. (1982a), Arnold V.I. et al (1988), the virtual displacement condition for a general nonholonomic constraint of the form

$$\phi^a(q, \dot{q}, t) = 0, a = 1, 2, \dots, k$$

is obtained by variation of the generalized coordinates and the generalized velocities and is given by:

$$\frac{\partial \phi^a}{\partial q^A} \delta q^A + \frac{\partial \phi^a}{\partial \dot{q}^A} \delta \dot{q}^A = 0, a = 1, 2, \dots k \,. A = 1, 2, \dots . n \,. \qquad (3.1)$$

The trajectory equations are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \left(\frac{\partial L}{\partial q^A}\right) - R_A - Q_{nc} + \mu_a \left(\frac{d}{dt}\left(\frac{\partial \phi^a}{\partial \dot{q}^A}\right) - \left(\frac{\partial \phi^a}{\partial q^A}\right)\right) - \dot{\mu}_a \frac{\partial \phi^a}{\partial \dot{q}^A} = 0 \,, (3.2)$$

Where, Q_{nc} is non-conservative force and R_A is the constraint force.

When

$$R_A . \delta q^A = 0 , \tag{3.4}$$

then R_A will not appear in (3.2) and hence the equation becomes identical with that of vakonimic motion equation (2.11).But the

work done by the constraint forces vanishes for *ideal constraints* where we have smooth surfaces.
Let us investigate this as follows:

When constraint equation $\phi^a(q, \dot{q}, t) = 0$ is integrable, then there exists a function $g = g^a(q, t)$ such that:

$$\phi^a(q, \dot{q}, t) = \frac{dg(q, t)}{dt} = \frac{\partial g^a}{\partial q^A} \dot{q}^A + \frac{\partial g^a}{\partial t} = 0. \tag{3.5}$$

The integration of $\dfrac{\partial g^a}{\partial q^A} \dot{q}^A + \dfrac{\partial g^a}{\partial t} = 0$ represents a surface $g^a(q, \dot{q}, t) = c$, where c is a constant.

Now

$$\frac{d}{dt}\frac{\partial \phi^a}{\partial \dot{q}^A} - \frac{\partial \phi^a}{\partial q^A} = \frac{d}{dt}\left(\frac{\partial g^a}{\partial \dot{q}^A}\right) - \frac{\partial}{\partial q^A}\left(\frac{\partial g^a}{\partial q^A}\dot{q}^A + \frac{\partial g^a}{\partial t}\right) = 0$$

$$=> \frac{d}{dt}\left(\frac{\partial \phi^a}{\partial \dot{q}^A}\right) = \left(\frac{\partial \phi^a}{\partial q^A}\right) \tag{3.6}$$

The variation of the constraints, equation (3.1), becomes:

$$0 = \frac{\partial \phi^a}{\partial q^A}\delta q^A + \frac{\partial \phi^a}{\partial \dot{q}^A}.\delta \dot{q}^A = \frac{d}{dt}\left(\frac{\partial \phi^a}{\partial \dot{q}^A}\right)\delta q^A + \frac{\partial \phi^a}{\partial \dot{q}^A}.\delta \dot{q}^A$$

$$= \frac{d}{dt}\left(\frac{\partial \phi^a}{\partial \dot{q}^A}.\delta q^A\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}^A}\left(\left(\frac{\partial g^a}{\partial q^A}\dot{q}^A + \frac{\partial g^a}{\partial t}\right).\delta q^A\right)\right) = \frac{d}{dt}\left(\frac{\partial g^a}{\partial q^A}.\delta q^A\right)$$

$$\Longrightarrow \frac{d}{dt}\left(\frac{\partial g^a}{\partial q^A}.\delta q^A\right) = 0$$

From the equality:

$$\frac{d}{dt}\left(\frac{\partial \phi^a}{\partial \dot{q}^A}.\delta q^A\right) = \frac{d}{dt}\left(\frac{\partial g^a}{\partial q^A}.\delta q^A\right) = 0 , \tag{3.7}$$

integrating on equation on an interval [a, b] we obtain:

$$\frac{\partial \phi^a}{\partial \dot{q}^A} \cdot \delta q^A = \frac{\partial g^a}{\partial q^A} \cdot \delta q^A = 0. \tag{3.8}$$

From (3.8) one can infer that, the virtual displacement of variation of the constraints is tangential to the surface $g = g^a(q,t)$. This result is expected since for ideal constraints the surface is assumed to be perfectly smooth, the constraint forces are normal to the virtual displacements. Hence we can conclude that

in case of ideal and integrable constraint, the virtual work done by the constraint force, $R_A \cdot \delta q^A = 0$ .That is vakonomic mechanics (3.2) based on (3.6) and (3.8), and with the choice of $\lambda_a = -\dot{\mu}_a$ have the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \left(\frac{\partial L}{\partial q^A}\right) = Q_{nc} + \lambda_a \frac{\partial \phi^a}{\partial \dot{q}^A} . \tag{3.9}$$

We can conclude that**,** the vakonomic mechanics gives the same dynamic equation of motion as nonholonomic mechanics and Chetaevs method in the case of ideal constraints. In this case (3.2) reduces to equation (3.9).

Vakonomic Mechanics in the Case of Nonholonomic Constraints
We shall show a unified approach to nonhohlonomic systems and compare it with vakonomic mechanics.
Let us start from variations of the constraint equation (3.1). Assuming the end point condition:

$$\delta q^A|_{t=a} = \delta q^A|_{t=b} = 0 \quad , \tag{4.1}$$

and integrating equation (3.1) on $[a\ b]$ we obtain an integral constraint:

$$\int_a^b \left(\frac{d}{dt}\left(\frac{\partial \phi^a}{\partial \dot{q}^A}\right) - \frac{\partial \phi^a}{\partial q^A}\right)\delta q^A = 0.$$

Now given a functional

$$\mathcal{J}(\boldsymbol{q}(t)) = \int_a^b L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\, dt,$$

Subject to boundary conditions (4.1) and k first order nonlinear nonholonomic constraints of the form :

$$\phi^\alpha(q^A, \dot{q}^A) = 0, 1 \leq \alpha \leq k , \tag{4.2}$$

construct the extended functional:

$$J^*(q(t), \mu_\alpha) = \int_a^b (L(q, \dot{q}, t) \, d + \mu_\alpha \phi^\alpha(q, \dot{q}, t)) dt, \qquad (4.3)$$

and extremizing over $\{q^1, \ldots, q^n, \mu_1, \ldots, \mu_k\}$ leads to:

$$\int_a^b \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \left(\frac{\partial L}{\partial q^A} \right) - \mu_\alpha \left(\frac{d}{dt} \left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A} \right) - \frac{\partial \phi^\alpha}{\partial q^A} \right) \right) \delta q^A dt = 0, \ \forall q^A , \ (4.4)$$

Where,

$A = 1, \ldots, n. \ \alpha = 1, \ldots, k.$

On the other hand, multiplying (3.1) by the scalar function $\mu_a(t)$ and then

integrating it on [a b] with respect to time, taking into account (4.1), we obtain:

$$\int_a^b \mu_\alpha \left(\frac{d}{dt} \left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A} \right) - \frac{\partial \phi^\alpha}{\partial q^A} \right) \delta q^A \, dt = \int_a^b \dot{\mu}_\alpha \frac{\partial \phi^\alpha}{\partial \dot{q}^A} \delta q^A dt. \qquad (4.5)$$

Substituting (4.5) in (4.4) leads to:

$$\int_a^b \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \left(\frac{\partial L}{\partial q^A} \right) - \dot{\mu}_\alpha \frac{\partial \phi^\alpha}{\partial \dot{q}^A} \right) \delta q^A dt = 0. \qquad (4.6)$$

Making a proper choice of λ^α and substituting $\lambda^\alpha \triangleq \dot{\mu}_\alpha$, and including the constraint equations we have:

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \left(\frac{\partial L}{\partial q^A} \right) = \lambda^\alpha \frac{\partial \phi^\alpha}{\partial \dot{q}^A} \qquad \alpha = 1, \ldots, k. \\ \phi^\alpha(q, \dot{q}, t) = 0 \end{cases} \qquad (4.7)$$

Equation (4.7) gives a complete motion equation for systems constrained by first order nonlinear nonholonomic constraints of the form (4.2).

Note that (4.7) is the same as equation (2.9) including the constraint equations, except that Chetaev condition

(i.e. $\frac{\partial \phi^\alpha}{\partial \dot{q}^A} \cdot \delta q^A = 0$) is nowhere used in the proof. Hence, expression (4.7) is a unified motion equation of nonholonomic system for nonholonomic constraints of the form (4.2).

Remark: The Chetaev condition in the example of the rolling sphere is nonzero. $\frac{\partial \phi^\alpha}{\partial \dot{q}^A} \delta q^A \neq 0$. Indeed:

$$\frac{\partial \phi^\alpha}{\partial \dot{q}^A} \cdot \delta q^A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta y + \begin{pmatrix} 0 \\ a \end{pmatrix} \delta \varphi_1 + \begin{pmatrix} -a \\ 0 \end{pmatrix} \delta \varphi_2 \neq 0.$$

Let us compare vakonimic mechanics, equation (2.11), and our model (4.7).

I. For these two to be equal, the term $\frac{d}{dt}\left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A}\right) - \frac{\partial \phi^\alpha}{\partial q^A}$ in expression (2.11) has to vanish. We have already seen in equation (3.6) that $\frac{d}{dt}\left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A}\right) - \frac{\partial \phi^\alpha}{\partial q^A} = 0$, if the constraints are holonomic. That is (2.11) and (4.7) are equal provided that the constraints are holonomic.

II. As it is mentioned in definition2, vakonomic method of motion is obtained through a purely variational principle by imposing the fulfilling of the constraints *on the variations themselves*, not on the infinitesimal

variations as it is the case in nonholonomic mechanics.

In this paper, it is shown that the *variations of linear nonintegrable nonholonomic constraints* are not consistent with the constraints. Let us investigate this problem using *first order nonintegrable linear nonholonomic* velocity constraints. In this case the constraints can be expressed in the form of expression (2.5).

Suppose the variation of expression (2.5) satisfies the constraints. Then we have:

$$\delta\phi^\alpha = \phi^\alpha(q + \delta q, \dot{q} + \delta\dot{q}, t) - \phi^\alpha(q, \dot{q}, t) = 0.$$

Now with this immediate equation, imposing end point condition (4.1) and integrating over [a, b] for all δq^A, we obtain:

$$\frac{d}{dt}\left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A}\right) - \frac{\partial \phi^\alpha}{\partial q^A} = 0.$$

Moreover,

$$\frac{d}{dt}\left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A}\right) = \frac{d}{dt}\mu_{aA}(q,t) = \frac{\partial \mu_{aA}}{\partial q^i}\dot{q}^i + \frac{\partial \mu_{aA}}{\partial t}, \quad \frac{\partial \phi^\alpha}{\partial q^A} = \frac{\partial \mu_{aA}}{\partial q^j}\dot{q}^A + \frac{\partial \mu_a}{\partial q^j}.$$

Hence,

$$0 = \frac{d}{dt}\left(\frac{\partial \phi^\alpha}{\partial \dot{q}^A}\right) - \frac{\partial \phi^\alpha}{\partial q^A} = \left(\frac{\partial \mu_{aA}}{\partial q^i}\dot{q}^i - \frac{\partial \mu_{aA}}{\partial q^j}\dot{q}^A\right) + \left(\frac{\partial \mu_{aA}}{\partial t} - \frac{\partial \mu_a}{\partial q^j}\right)$$

$$= \left(\frac{\partial \mu_{aj}}{\partial q^i} - \frac{\partial \mu_{ai}}{\partial q^j}\right)\dot{q}^i + \left(\frac{\partial \mu_{aA}}{\partial t} - \frac{\partial \mu_a}{\partial q^j}\right)$$

From this last equation, we see that for the variation of the constraints, $\delta\phi^\alpha$, to

satisfy the constraint equation, the constraint equation (2.5) must be *an exact differential equation*, i.e.

$$\left(\frac{\partial \mu_{aj}}{\partial q^i} - \frac{\partial \mu_{ai}}{\partial q^j}\right)\dot{q}^i + \left(\frac{\partial \mu_{aA}}{\partial t} - \frac{\partial \mu_a}{\partial q^j}\right) = 0, \qquad\qquad (4.8)$$

holds true, if and only if (2.5) is exact. Provided that the linear constraints (2.5) satisfy condition (4.8) (i.e. it is an exact differential equation), an integrand function of the form $g^a(q,t)$ therefore exists but may be unknown. Such constraints are termed *semi-holonomic,* Goldstein H. (2001). Semi-holonomic constraints are holonomic constraints but their integrand may not be known. The existence of such a function, $g^a(q,t)$ granted by exactness condition (4.8), contradicts our assumption that the constraint is nonintegrable.

We can now conclude that, the immediate above proof and the discussion there in,

$$\dot{\omega} = adq^1 + bdq^2 + cdq^3 = 0 \tag{4.9}$$

Where $a, b,$ and c are functions of (q^1, q^2, q^3) of class C^1 and the Pfaffian form $\dot{\omega}$ does not admit an

ascertains that, only the variation of integrable constraints (semi-holonomic and holonomic constraints) satisfy the constraint equations of a given system. In a nonholonomic system, only the original constraint satisfies the constraint equations and its variations are not consistent with the constraints. To strengthen this conclusion let us take up the following particular example.

Example 2
Consider a linear nonholonomic constraint below and suppose a particle is moving in the space, subject to the constraint:

integrating factor. That is ω is not known. The original orbit surely satisfies constraint (4.9), and so by hypothesis do the velocity and variations from it satisfies (4.9). So we have:

$$a\dot{q}^1 + b\dot{q}^2 + c\dot{q}^3 = 0 \tag{4.10}$$
$$a\delta q^1 + b\delta q^2 + c\delta q^3 = 0 \tag{4.11}$$

Suppose the *variation* satisfies the constraint condition. That is possible only if

$$\delta(\dot{\omega}) = \delta(a\dot{q}^1 + b\dot{q}^2 + c\dot{q}^3) = 0 \tag{4.12}$$

From (4.11) at each instant of time we have:

$$\frac{d}{dt}(\delta\omega) = \frac{d}{dt}(a\delta q^1 + b\delta q^2 + c\delta q^3) = 0 \tag{4.13}$$

Note that we use the commutation: $\delta\left(\frac{dq}{dt}\right) = \frac{d}{dt}(\delta q)$ for a linear in velocity nonholonomic systems, Zhongheng G. et al (1989). Subtracting (4.12) from (4.13) we have:

$$\frac{d}{dt}(\delta\omega) - \delta\left(\frac{d\omega}{dt}\right) = \left(\frac{da}{dt}\delta q^1 - \dot{q}^1\delta a\right) + \left(\frac{db}{dt}\delta q^2 - \dot{q}^2\delta b\right) +$$
$$\left(\frac{dc}{dt}\delta q^3 - \dot{q}^3\delta c\right) = 0. \tag{4.14}$$

Expanding equation (4.14) we obtain:

$$\left(\frac{\partial c}{\partial q^2} - \frac{\partial b}{\partial q^3}\right)(\dot{q}^2\delta q^3 - \dot{q}^3\delta q^2) + \left(\frac{\partial a}{\partial q^3} - \frac{\partial c}{\partial q^1}\right)(\dot{q}^3\delta q^1 - \dot{q}^1\delta q^3) + \left(\frac{\partial b}{\partial q^1} - \frac{\partial a}{\partial q^2}\right)(\dot{q}^1\delta q^2 - \dot{q}^2\delta q^1) = 0. \qquad (4.15)$$

On the other hand, since the following proportion follows:

$$(a, b, c) \times [(\delta q^1, \delta q^2, \delta q^3) \times (\dot{q}^1, \dot{q}^2, \dot{q}^3)] = 0$$
$$\frac{(\dot{q}^2\delta q^3 - \dot{q}^3\delta q^2)}{a} = \frac{(\dot{q}^3\delta q^1 - \dot{q}^1\delta q^3)}{b} = \frac{(\dot{q}^1\delta q^2 - \dot{q}^2\delta q^1)}{c}.$$

From which equation (4.15) becomes:

$$a\left(\frac{\partial c}{\partial q^2} - \frac{\partial b}{\partial q^3}\right) + b\left(\frac{\partial a}{\partial q^3} - \frac{\partial c}{\partial q^1}\right) + c\left(\frac{\partial b}{\partial q^1} - \frac{\partial a}{\partial q^2}\right) = 0. \qquad (4.16)$$

This tells us that the constraint is exact and hence integrable. This is not true since the constraint is supposed to be nonintegrable. The assumption that the variation satisfies the constraints led to a contradiction.

Hence, the variations from the original nonholonomic constraint do not conform to the equations of the constraint. In other words, there is no varied path C_s that satisfies the constraint condition in a linear nonholonomic system. It needs to be noted that the infinitesimal variations X and the actual curve C (Look fig.1) satisfies the constraint conditions.

According to this paper, the main reason for vakonomic mechanics to give different motion equations to nonholonomic mechanics and our model equation (4.7) is that, in a purely variational approach to mechanical modeling, the variations from the original nonholonomic constraints don't conform to the original constraint equations.

As we have shown, for holonomic constraints all the motion equations obtained from Vakonomic nonholonomic mechanics and equation (4.7), Chetaev equation without Chetaev condition, are the same.

CONCLUSION:

In this article, different methods of modeling mechanical systems were investigated. The Vakonomic mechanics is compared against the nonholonomic mechanics and Chetaev method of constructing motion equations of constrained mechanical systems. It is ascertained that all of them gives the same motion equations for the case of holonomic constraints and the Vakonomic mechanics is different from the remaining ones for nonholonomic systems. The reason why Vakonomic mechanics is different in the case of nonholonomic constraints is detailed in the paper. Moreover, Chetaev equation is proved without using its conditions. It is verified using an example that there are cases when Chetaev condition is not valid.

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